Daily Quiz Items By Date

The daily quiz and many of the exam problems will be taken from the following list. The problems for a particular quiz are listed under that date in the first section.

This document will be updated throughout the course. It was last updated December 8, 2015

1 Numbers, Variables, Polynomials, Binomial Theorem, Linear Equations, Aryabhata Algorithm

1.1 August 26

1. What is the difference between a number and a numeral?
   **Ans:** A numeral is a name for a number.

2. Assume that $3 = 4$ and prove that you are the president of UK.
   **Ans:** If $3 = 4$ then subtracting 2 from both sides we get $1 = 2$. The president of UK and you are 2 people but $2 = 1$ so the president of UK and you are one person. Therefore you are the president of UK.

3. What property do the real numbers have that the complex numbers do not?
   **Ans:** Order. If there is order then non-zero squares must be positive. So $1^2 = 1 > 0$ and $i^2 = -1 > 0$. But the sum of positive numbers must also be positive which would say that $1 + (-1) > 0$ which would say $0 > 0$.

4. **Definition:** A numeral is a name for a number.

1.2 August 28

1. Write the number represented by the decimal $0.272727\ldots$ as the ratio of two integers.
   **ANS:** $\frac{27}{99} = \frac{3}{11}$

2. Show that $0.9999\ldots = 1$
   **ANS:** Let $x = 0.9$. Then $10 * x = 9 + x$ so $x = 1$

3. Express $\frac{23}{17^2} + \frac{51}{33}$ in the form $\frac{a}{b}$ where $a$ and $b$ are integers. Your answer must be exact but need not be in lowest terms.
   **Ans:** $\frac{23}{17^2} + \frac{51}{33} = \frac{23*33+17^2*51}{17^2*33} = \frac{15498}{9537}$
4. If \( A = (3, 14) \) and \( B = (8, 2) \) are points in the plane. Use the distance formula to calculate the distance from \( A \) to \( B \) is exactly.

**Ans** \( \sqrt{(8 - 3)^2 + (2 - 14)^2} = 13. \)

5. **Definition:** Decimal numbers are rational numbers that can be written in the form \( \frac{a}{10^n} \) where \( a \) is an integer and \( n \) is a non-negative integer.

6. Suppose \( x \) is a real number. According to the definition the absolute value of \( x \) is

\[
|x| = \begin{cases} 
  x & \text{if } x > 0 \\
  -x & \text{if } x \leq 0
\end{cases}
\]

**Ans:**

1.3 **August 31**

1. Use the axioms for: additive identity, additive inverse, associative addition, and the distributive law to explain why \( a * 0 = 0 \) for any number \( a \).

**Ans:**

\[
\begin{align*}
0 + 0 &= 0 \\
a * 0 + a * 0 &= a * 0 \\
(a * 0 + a * 0) + -a * 0 &= a * 0 + -a * 0 \\
a * 0 + (a * 0 - a * 0) &= a * 0 + -a * 0 \\
a * 0 + (0) &= 0 \\
a * 0 &= 0
\end{align*}
\]

2. Write down the decimal expansion of a non-rational number and explain why the number it represents is not rational.

**Ans:** For example write \( x = .11010010001000010000010000010000000001 \cdots \) begins with 1 followed by 0 zeroes, then 1 followed by 2 zeroes, etc. This cannot be rational since the rational numbers have decimal expansions that eventually begin to repeat. If this one had a pattern that repeated every \( n \) digits the a complete pattern would have to lie in every consecutive block of length \( 2n \). Since the \( x \) has blocks zeroes of length \( 2n \) that would say that the repetition pattern is all zeroes which would make the decimal expansion 0 from some point on which would be a contradiction since for any \( M \) there is a 1 in the expansion beyond the \( M^{th} \) place.

1.4 **September 2**

1. Explain why the number with decimal expansion 1234567891011121313\ldots100101102\ldots (every integer occurs in sequence) cannot be rational.

**Ans.** If this were rational then there is a pattern \( P \) of length \( N \) for some \( N \) such that from some point on the expansion is \( \ldots PPPPPP \ldots \). If \( p \) is the digit at which the pattern starts to repeat then \( 10^{p+2N} \) must occur after the \( p^{th} \) position and it has more than \( 2N \) zeroes. So an entire pattern \( P \) would have to fall into this block of zeroes. Therefore that block \( P \) would itself be all zeroes. This says that the expansion is 0 from that point on which is false. So the assumption that \( x \) has a repeating decimal leads to a contradiction and therefore that assumption is false.
2. What are the quotient and remainder that would be produced if we were to divide $7^{100} + 17$ by $7$ using long division? Explain your answer.

**Ans:** The quotient $q$ and remainder $r$ are the unique integers which satisfy $7^{100} + 17 = 7q + r$ with $0 \leq r < 7$.

We observe that $N = (7^{100} + 17) = 7^{100} + 10 + 7 = 7^{100} + 7 + 3 + 7 = 7^{100} + (2)7 + 3 = 7(7^{99} + 2) + 3$. So since $0 \leq 3 < 7$, 3 is the remainder since there is only one way to write $N = 7q + r$ with $0 \leq r < 7$.

3. What are the quotient and remainder that would be produced if we were to divide 26,423 by 89 using long division? Explain your answer.

**Ans:** From the calculator $\frac{26423}{89} = 296.8876404$. This says that the quotient $q = 296$. The remainder is $26,423 - 296(89) = 79$.

4. Explain why the number

$$x = .1 10 100 1000 10000 100000 1000000 \cdots$$

(every power of 10, in sequence) cannot be rational.

**Ans.** If the number were rational then from some point on it would repeat in a pattern $P$ of length $n$. There are blocks of 0’s of all lengths in the expansion for $x$ so in particular there are blocks of 0’s of length $2n$. Any such block would have to contain a complete pattern which means that $P$ would have to be all 0’s. That would mean that the expansion for $x$ would be all 0’s from some point on but after every place there is always another 1 so it is impossible for $x$ to be rational.

5. What is the remainder if 123,677,339,876,907,100,234 is divided by 10,000?

**Ans:** $123,677,339,876,907,100,234 = 123,677339876901, 710*10,000 + 234 = 10,000Q + 234$. With $0 \leq 234 < 10,000$. Since the quotient and remainder are unique, 234 must be the remainder.

### 1.5 September 4

1. Express $\frac{\frac{1}{1+\frac{2}{3}}}{\frac{1}{1+\frac{2}{3}}}$ in the form $\frac{a}{b}$ where $a$ and $b$ are integers.

**Ans:** $\frac{1}{1+\frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$

2. Determine whether $\frac{12399}{68799} = \frac{12299}{67899}$. Show your work. Your solution cannot involve decimal numbers. Multiplying, adding, and subtracting integers with the calculator is ok.

**Ans:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ so these are equal if and only if $12399 \times 67899 = 12299 \times 68799$. This is 841879701 = 846158901 which is false so the two are not equal.

3. Determine whether $\frac{\frac{3+2i}{5+i}}{\frac{5-i}{4+6i}}$. Explain your answer!!

**Ans:** False because $(3+2i)(4+6i) \neq (5+i)(5-i)$
1.6 September 9

1. If \( z = 3 + 4i \) then

   (a) write \( z = ru \) where \( r > 0 \) and \( |u| = 1 \).

   \textbf{Ans:} \( z = \sqrt{3^2 + 4^2} \left( \frac{3}{\sqrt{3^2 + 4^2}} + \frac{4}{\sqrt{3^2 + 4^2}} i \right) = 5 \left( \frac{3}{5} + \frac{4}{5} i \right) \)

   (b) If the \( u \) in part a) is expressed as \( u = \cos(\theta) + i \sin(\theta) \) with \( 0 \leq \theta < 2\pi \) then, in radians, \( \theta = \tan^{-1} \left( \frac{4}{3} \right) = .9272952179 \) (from calculator)

2. \textbf{Definition: A Linear Equation} of (or in) the variables \( x_1, \ldots, x_n \) is an expression of the form \( L_1(x_1, x_2, \ldots, x_n) = L_2(x_1, x_2, \ldots, x_n) \) where each \( L_i \) is a polynomial of degree at most 1 in each of its variables. That is, each \( L_i \) can be written in the form \( \alpha_1 x_1 + \cdots + \alpha_n x_n - \beta \).

3. \textbf{Definition} A linear equation is \textbf{consistent} if it has at least one solution.

4. \textbf{Definition} A linear equation is \textbf{inconsistent} if it has no solutions.

5. \textbf{Definition} To \textbf{solve} a linear equation is to determine whether it is consistent and, if it is, to find all of its solutions.

6. Give an example of a linear equation in the variables \( x, y \) that has an infinite number of solutions. Explain your answer.

   \textbf{Ans:} Any linear equation in two variables that is not inconsistent will have an infinite number of solutions. For instance \( x - y = 0 \) has \((a, a)\) as a solution for every real number \( a \).

7. Give an example of a linear equation in the variables \( x, y \) that is inconsistent.

   \textbf{Ans:} For example \( x + y = x + y + 1 \) is inconsistent since it simplifies to \( 0 = 1 \) which has no solutions. (Any linear equation that is inconsistent is going to have to simplify to an equation of the type \( 0 = a \) where \( a \neq 0 \).)

8. Give an example of a linear equation in the variable \( x \) that has exactly one solution.

   \textbf{Ans:} For example \( x = 1 \).

9. \textbf{Definition: S is a system of equations} in the variables \( \{x_1, x_2, \ldots, x_n\} \) if \( S = \{E_1, E_2, \ldots, E_m\} \) where each \( E_i \) is an equation in the variables \( \{x_1, x_2, \ldots, x_n\} \).

10. \textbf{Definition:} If \( S = \{E_1, E_2, \ldots, E_m\} \) is a \textbf{system of equations} in the variables \( \{x_1, x_2, \ldots, x_n\} \) then \((s_1, s_2, \ldots, s_n)\) is a solution to the system \( S \) if it is a solution to \( E_i \) for each \( i \).
11. Explain why \((1, -1)\) is a solution to the system of equations \(S = \{E_1 : x + 3y = -2, E_2 : x - y = 2\}\)

**Ans:** 
\(E_1 : 1 + 3(-1) = -2\) is true  
\(E_2 : 1 - (-1) = 2\) is true

12. (This problem was edited on Sept 9 to correct a typo.)

\(S = \{x - 3y = 6, 2x + 7y = -1\}\) is a system of linear equations in the variables \(x, y\). Solve \(S\). Show your work.

**Ans:**  
\[
\begin{align*}
    x - 3y &= 6 \\
    2x + 7y &= -1
\end{align*}
\]

\(E_2 \rightarrow E_1 - 2E_1 \quad x - 3y = 6 \quad 0 - y = 1 \)

\(y = -1, \quad x - 3(-1) = 6, \quad x = 3. \) The only solution is \((x, y) = (3, -1)\)

### 1.7 September 11

1. **Definition:** The constant, \(c\), in the monomial \(M(x) = c \cdot x^n\) is called the **coefficient** of \(M\).

2. **Definition:** The **degree** of a monomial \(M(x) = c \cdot x^n\) is \(n\) if \(c\) is not 0. The zero monomial does not have a degree. If \(M\) is a monomial in \(m\) variables then the degree of \(M\) is the ordered \(m\)-tuple of exponents of each of the variables.

3. **Definition:** A **polynomial** is a sum of finitely many monomials.

4. A polynomial is **simplified** if it is 0 or is written as a sum of monomials, no two of which have the same degree. In the latter case, the monomials are called the **terms** of the polynomial.

5. **Definition:** The degree of the polynomial \(f(x)\) is the largest of the degrees of its terms that have a non-zero coefficient.

6. **Definition:** Two polynomials are **equal** if and only if their terms of the same degree are equal when they are in a simplified form.

7. (This problem was edited on Sept 10 to correct a typo.)

Suppose \(f(x), g(x), \) and \(h(x)\) are polynomials such that

(a) \(f\) has degree 5  
(b) \(g\) has degree 7  
(c) \(h\) has degree 10.

Answer the following with properly justified responses.

(a) The degree of \(f \cdot h\) is 
(b) The degree of \(f(12 \cdot g - 11 \cdot h)\) is 
(c) If \(g(x)\) is divided by \(f(x)\), using long division to write \(g(x) = f(x) \cdot q(x) + r(x)\) then, assuming it is not zero the possible degrees for \(r(x)\) are and the possible degrees of \(q(x)\) are 

**Ans:** (a) 15, (b) 50, (c) \{0, 1, 2, 3, 4\} (d) 2
8. (This problem was edited on Sept 10 to correct a typo.)

Fill in the blanks; justify your answers.

(a) If \( f(x) = x^5 - 2x^4 - 47x^3 + 116x^2 + 560x - 1600 \) then the coefficient of \( x^5 \) in \((x - 2)f(x)\) is _________.

(b) If \( f(x) = x^5 - 2x^4 - 47x^3 + 116x^2 + 560x - 1600 \) and \((x - a)f(x) = (x - 5)(x^5 - 2x^4 - 47x^3 + 116x^2 + 560x - 1600)\) then \( a = \) _________.

Ans: (a) \(-4\), (b) \(5\)

9. (This problem was edited on Sept 11 to correct a typo.)

Suppose \( f(x) = x^2 + 3x + 6 \). Determine the value of \( s \) such that \( f(u + s) = u^2 + C \) where \( C \) is a constant. In other words, in the expansion of \( f(u + s) \), the coefficient of \( u \) is zero. \( s = \) _________.

Justify your answer.

With your value of \( s \) the value of \( C \) is _________. Justify your answer.

Ans: \(-\frac{3}{2}, \frac{15}{4}\)

10. If \((4x - q)^7\) is expanded and simplified then the coefficient of \( x^5q^2 \) will be _________. Justify your answer.

Ans: Coefficient of \( \left(\frac{7}{5}\right)(4x)^5(-q)^2 = 21504\)

11. If \( f(x) = 3x^4 - 7x^3 + 2x - 9 \), \( g(x) = x^3 - 5x^2 - x - 11 \), and \( f(x)g(x) = 3x^7 - 22x^6 + Ax^5 + 42x^4 - 96x^3 + 43x^2 + 31x - 99 \) then \( A\) (the coefficient of \( x^5 \) in \( f(x)g(x)\)) is _________.

Ans: \(32\)

12. Calculate the degree and lead coefficient of \((x + 3)^{16} - (x + 4)^{16}\). Justify your answers. Degree = _________. Lead coefficient = _________.

Ans: \(15, -16\)

1.8 September 14

1. If \((x - 1)^{500}\) is expanded and simplified then the coefficient of \( x^{498} \) will be _________. You may leave your answer in factored form. (Show your work!)

Ans: \(\binom{500}{498} = \frac{500 \cdot 499}{2} = 250 \cdot 499\)

2. Suppose \( t \) is a number such that \( 3 \cdot (t - 7)^2 - 5 \cdot (t - 7) = -2 \). Then \( t = \) _________.

Ans. \( x = t - 7 \) then \( 3x^3 - 5x + 2 = 0 \). By quadratic formula \( x = \frac{-5 \pm \sqrt{(-5)^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} \). \( x = 1, 2, 3 \frac{2}{3} \)

\( t = x + 7 = 8, 7, 3 \frac{2}{3} \)
3. If \((3y - 2x)^5\) is expanded and simplified then the coefficient of \(x^2y^3\) will be \(\underline{\underline{\frac{15}{4}}}\). You may leave your answer in factored form. (Show your work!)

**Ans:** \(C(5, 3)(3y)^3(-2x)^2 = (10)3^3(-2)^2x^2y^3\)

4. If you were to write out Pascal’s triangle to the row for \(n = 20\) (i.e. the one that begins 1 20 190 1140 ...) then the sum of all of the numbers on that row (i.e. 1 + 20 + 190 + 1140 + ...) is \(\underline{\underline{\frac{15}{4}}}\).

The alternating sum 1 - 20 + 190 - 1140 + ... Explain your answer.

**Ans:** \((1 + 1)^{20} = 2^{20}, (1 - 1)^{20} = 0^{20} = 0\)

### 1.9 September 16

1. **Definition:** A **Linear Equation** in the variables \(x_1, \ldots, x_n\) is an equation \(L_1 = L_2\) in the variables \(x_1, \ldots, x_n\) where each monomial in the polynomials \(L_1\) and \(L_2\) has the form \(a_1x_1 + \cdots + a_nx_n = b\).

2. **Definition:** A Linear Equation is **homogenous** if it can be written in the form \(a_1x_1 + \cdots + a_nx_n = 0\).

3. **Definition:** A **system of linear equations** or a **linear system** is a set of linear equations. A linear system is **homogenous** if every member of the system is a homogenous linear equation.

4. **Definition:** A **solution** to a linear equation \(a_1x_1 + \cdots + a_nx_n = b\) is an ordered list of scalars \((s_1, s_2, \ldots, s_n)\) such that the equation becomes a true statement when \(s_i\) is substituted for \(x_i\) for each \(i = 1, \ldots, n\). An ordered list \((s_1, s_2, \ldots, s_n)\) is a solution to the linear system \(S\) if it is a solution to each linear equation in \(S\).

5. Solve the system of equations:

\[
\begin{align*}
4x + 10y &= 9 \\
-3x - 4y &= 1
\end{align*}
\]

by Cramers Rule. Express each of \(x\) and \(y\) as the ratio of determinants of matrices and calculate each of the individual determinants. Show your work! \(x = \underline{\underline{\frac{46}{14}}}, y = \underline{\underline{\frac{31}{14}}}\).

**Ans:** \(x = \frac{46}{14}, y = \frac{31}{14}\)

6. The height and width of a rectangle are unknown. When 12 exact copies are arranged in 4 stacks of three to make a larger rectangle then the perimeter of the larger rectangle is 80 units. When that same 12 are arranged in 6 stacks of 2 then the perimeter of the larger rectangle is 100 units.
What are the dimensions of the rectangle? \(\text{width} = \underline{\quad}, \text{height} = \underline{\quad}\).

**Ans:** \(\text{width} = 7, \text{height} = 4\).

7. A grocery shelf contains cans of red beans and cans of black beans. All cans of red beans have the same weight and all cans of black beans have the same weight. A shopper purchases 4 cans of red beans and 4 cans of black beans and her purchase weighs a total of 28 pounds. Another shopper purchases 2 cans of red beans and 3 cans of black beans and his purchase weighs a total of 19 pounds. Determine the weight (in pounds) of each can of red beans and of each can of black beans. Justify your answers.

One can of red beans weighs \(\underline{\quad}\) and one can of black beans weighs \(\underline{\quad}\).

**Ans:** 2 pounds and 5 pounds.

### 1.10 September 18

1. Complete the Aryabhata table and use the result to answer the following questions.

\[
\begin{array}{ccc}
1 & 0 & 14 \\
0 & 0 & 1 & 32 \\
\end{array}
\]

**Ans**

\[
\begin{array}{ccc}
1 & 0 & 14 \\
0 & 0 & 1 & 32 \\
-2 & 1 & 0 & 14 \\
-3 & -2 & 1 & 4 \\
-2 & 7 & -3 & 2 \\
-16 & 7 & 0 \\
\end{array}
\]

(a) The greatest common divisor (GCD) of 14 and 32 is \(\underline{\quad}\).

**Ans:** 2

(b) The least common multiple (LCM) of 14 and 32 is \(\underline{\quad}\).

**Ans:** \(16(14) = 7(32) = 224\)

2. Find an integer \(n\) such that if \(n\) is divided by 17 then the remainder is 3 and if \(n\) is divided by 11 then the remainder is 7. You may use the following Aryabhata table. Note there is more than one answer. You only need to provide one.
Ans: We want $n$ so that $n = 17q_1 + 3$ and $n = 11q_2 + 7$.
This says we want $17q_1 + 3 = 11q_2 + 7$ or $17q_1 - 11q_2 = 7 - 3$
We want $17q_1 - 11q_2 = 7 - 3$
From the table $2*17-3*11 = 1$, $7-3 = 4$ so $4(2*17-3*11) = 4(1) = 7-3$, $4*2*17+3 = 4*3*11+7$.

So if we take $n = 4 * 2 * 17 + 3$ then the remainder on division by 17 is 3 while we also have $n = 4 * 3 * 11 + 7$ so the remainder on division by 11 is 7.
So 139 is one such number.

3. A rectangular room has red tiles of 36 inches lain end to end along one wall and blue tiles of length 34 inches lain end to end along the opposite wall. What is the smallest possible length of the room?

Ans The length of the room is a multiple of both 36 and 34 so it is a common multiple of 36 and 34. So the smallest length is $LCM(36, 34)$.
From the following Aryabhata table we see that $LCM(36, 34) = 36 * 17 = 34 * 18 = 612$. The smallest length of the wall is 612 in. = 51 ft.

4. Use the Aryabhata algorithm to express the fraction $\frac{9723}{10186}$ as the ratio of two positive integers with no common factor (i.e. in “lowest terms”).

Ans: From we see that $21 * 10186 = 22 * 9723$ so $\frac{21}{22} = \frac{9723}{10186}$

5. Fill in the missing entries in the following Aryabhata Algorithm table.

<table>
<thead>
<tr>
<th>Quotients</th>
<th>Answer 1</th>
<th>Answer 2</th>
<th>Integers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td>1</td>
<td>0</td>
<td>245</td>
</tr>
<tr>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>168</td>
</tr>
<tr>
<td>$-2$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-11$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>End</td>
<td>$-24$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The greatest common divisor (GCD) of 245 and 168 is 

From the table we can write GCD as a linear combination of 245 and 168.

\[ GCD = \_ \text{ } \_ \times 245 + \_ \text{ } \_ \times 168 \]

The least common multiple (LCM) of 245 and 168 is ___

Ans:

<table>
<thead>
<tr>
<th>Quotients</th>
<th>Answer 1</th>
<th>Answer 2</th>
<th>Integers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td>1</td>
<td>0</td>
<td>245</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>168</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>-1</td>
<td>77</td>
</tr>
<tr>
<td>-5</td>
<td>-2</td>
<td>3</td>
<td>14</td>
</tr>
<tr>
<td>-2</td>
<td>11</td>
<td>-16</td>
<td>7</td>
</tr>
<tr>
<td>End</td>
<td>-24</td>
<td>35</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) 7 (b) 11, -16 (c) 25 \times 245 = 35 \times 168 = 5880

6. A merchant has a balance, a large collection of 37 gram cylinders and another large collection of 29 gram cylinders. He also has an object which is supposed to weigh exactly 1 gram and he wants to check the weight. Study the following Aryabhata table and use it to find at least one way of checking the weight.

<table>
<thead>
<tr>
<th>Quotients</th>
<th>Answer 1</th>
<th>Answer 2</th>
<th>Integers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Begin</td>
<td>1</td>
<td>0</td>
<td>37</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>29</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>-5</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>-7</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>11</td>
<td>-14</td>
<td>1</td>
</tr>
<tr>
<td>End</td>
<td>-29</td>
<td>37</td>
<td>0</td>
</tr>
</tbody>
</table>

He checks the weight by putting the 1 gram object in the right pan as shown and balancing the pans by adding some of his cylinders to the pans as follows.

- He places _____ cylinders of weight _____ grams in the left pan and
- He places _____ cylinders of weight _____ grams in the right pan

Ans: Left pan: 11 cylinders of weight 37 gms and Right pan: 14 cylinders of weight 29 gms

(b) Find another solution. He could simply add any collection of the weights in one pan and add an exact copy to the other. If he wants to have only one kind of weight in each pan then he could add the least common multiple weight to each pan. That is he could add 29 of the 37 gm wts to the left pan and 37 of the 29 gm weights to the right.
1.11 September 21

1. Find all the points \( P(x, y) \) on the \( y \)-axis which are \( \sqrt{29} \) units away from the point \( A(-5, -8) \).
   Since \( P \) is on the \( y \)-axis its \( x \)-coordinate is 0.
   The \( y \)-coordinates of the points are __________
   Ans: \( d(A, P) = \sqrt{29}, (-5 - 0)^2 + (-8 - y)^2 = (\sqrt{29})^2 \)
   \( 25 + 64 + 16y + y^2 = 29 \), \( y^2 + 16y + 60 = 0 \), \( (y + 10)(y + 6) = 0 \), \( y = -10, y = -6 \)

2. Suppose \( \frac{ax+b}{cX-d} = \frac{2}{3} \). Express \( X \) as a formula in \( a, b, c, d \). You do not need to simplify.
   Ans: \( X = \frac{-3b+2d}{3a-2c} \)

3. Write \( \frac{x-1}{x+1} - \frac{x-2}{x+2} \) in the form \( \frac{A}{B} \) where \( A \) and \( B \) are polynomials. Show your work. You do not need to simplify the numerator or the denominator.
   Ans: \( \frac{2x}{(x+1)(x+2)} \)

4. If \( 7x^2y^3 \) is regarded as a polynomial in \( y \) with coefficients polynomials in \( x \) then its degree is __________ and its coefficient is __________.
   Ans: 3, 7\( x^2 \)

5. A rectangle has an unknown height, \( x \), and a base which is 5 units longer than the height.
   Express the area \( A \) and the perimeter \( P \) of the rectangle as a polynomials in \( x \).
   (b) If the area of the rectangle is 24 square units then the height of the rectangle is __________.
   (c) If the perimeter of the rectangle is 50 units then the height of the rectangle is __________.
   \( A(x) = \) __________. \( P(x) = \) __________.
   Ans: \( A(x) = (x+5)x, P(x) = 2x + 2(x+5) \), (b) \( x(x+5) = 24, x = 3, -8 \) so the height is 3 (c) \( 2x + 2(x+5) = 50 \), \( 4x = 40 \), \( x = 10 \).

6. A triangle and a 5 by 8 rectangle are cut out of a square of unknown side \( x \), leaving the shaded region indicated in the diagram.
Express the area $A$ of the region as a polynomial in $x$. 

**Ans:** $A(x) = \frac{1}{2}x^2 - 40$

7. If $x$ is any number then there is one and only one number $-x$ such that $x + -x = 0$. Use this fact to explain why $(-1)(-1) = 1$

**Ans:**

$(-1) * (-1 + 1) = -1 * 0 = 0$
$(-1)^2 + (-1) * 1 = 0$
$(-1)^2 + (-1) = 0$
$((-1)^2 + (-1)) + 1 = 0 + 1$
$(-1)^2 + ((-1) + 1) = 1$
$(-1)^2 + (0) = 1$
$(-1)^2 = 1$

8. Explain, using the number 5.347 that any number with a terminating decimal expansion has two different decimal expansions.

**Ans:** Consider . Then $10^3 X = 5346.\overline{9} = 5346 + \overline{9} = 5347$. So $X = \frac{5347}{10^3} = 5.347 = 5347$. So $X == 5.346999\overline{9}$ and $X == 5.347$
1.12 Additional Exam 1 Review Problems

This is the beginning of a Supplemental Set of Exam 1 Review Problems. Additional review problems will be added as we proceed toward the test on Sept 22. This page was December 8, 2015.

1. Find all solutions to the following system or show that it is inconsistent.

\[
\begin{align*}
2x - 3y &= 5, \\
6x + 7y &= 1
\end{align*}
\]

Ans: \(x = \frac{19}{16}, y = \frac{-7}{8}\)

2. Find all solutions to the following system or show that it is inconsistent.

\[
\begin{align*}
3x - 4y &= 3, \\
-6x + 8y &= 4
\end{align*}
\]

Ans: inconsistent

3. Find all solutions to the following system or show that it is inconsistent.

\[
\begin{align*}
x + y + z &= 2, \\
y + z &= 3, \\
z &= 4
\end{align*}
\]

Ans: \(\{x = -1, y = -1, z = 4\}\)

4. Fill in the blanks with rational numbers (i.e. fractions) (no decimals).

If \(A\) is the complex number \(2 - 3i\) then \(\frac{1}{A} = \underline{\underline{\quad}} + \underline{\underline{\quad}}i\)

Ans: \(\frac{2}{13} + \frac{3}{13}i\)

5. When a group of coins is arranged in groups of 7 there are 2 left over. When they are arranged in groups of 9 there are 6 left over. One possibility for the number of coins is ________.

\[
\begin{array}{ccc}
1 & 0 & 9 \\
-1 & 0 & 1 \\
-3 & 1 & -1 \\
-2 & -3 & 4 \\
7 & -9 & 0
\end{array}
\]

Ans: \(7x + 2 = 9y + 6\)
7x + 2 = 9y + 6
7x - 9y = 4

From table 
-3(9) + 4(7) = 1
-12(9) + 16(7) = 4 = 6 - 2
16(7) + 2 = 12(9) + 6 = 104

6. If $(3 - 2x^2)^{11}$ is expanded completely and put in standard form, what are the powers of $x$ for which the coefficient is not zero? _______

Ans: Even powers from 0 to 22

7. \{3x + 2y = -5, 5x + 4y = -13\} is a system of two linear equations in the variables $x$ and $y$. Use Cramer’s Rule to determine $x$. You do not need to calculate $y$. You must use Cramer’s Rule!

Ans:

8. If $f(x) = -2x^3 + 5x^2 + 3, g(x) = 5x^5 - 7x^4 + 2x - 11$, and $f(x)g(x) = -10x^8 + 39x^7 + Ax^4 + 32x^3 - 35x^6 - 55x^2 + 15x^5 + 6x - 33$.

Then $A = _______

Ans:

9. If $(3y - 2x)^5$ is expanded and simplified then the coefficient of $x^3y^2$ will be _______

Ans:

10. Suppose $f(x)$ is a polynomial of degree 5, $p(x)$ is a polynomial of degree 6, $g(x)$ is a polynomial of unknown degree, and $h(x)$ is a polynomial of degree 7.

(a) If $f(x)g(x) = p(x)^3h(x)^2$ then then degree of $g(x)$ is _______

Ans:

(b) The degree of $9x^2 - (3x - 7)^2$ is _______

Ans:

11. The remainder when $x^3 + x^2 + 1$ is divided by $x^2 - x + 2$ is _____.

Ans: $-3$

12. If 1,001,002,003,004,005 is divided by 10 the remainder is _____.

Ans: 5
13. √2 is a root of \(x^2 - 2\). If \(f(x) = x^5 + 4x^3 + 1\), write \(f(\sqrt{2}) = (\sqrt{2})^5 + 4(\sqrt{2})^3 + 1\) in the form \(a + b(\sqrt{2})\) where \(a\) and \(b\) are integers.

Ans: \(x^5 + 4x^3 + 1 = (x^2 - 2)(x^3 - 6x) + (1 + 12x)\)

\(\sqrt{2}^5 + 4(\sqrt{2})^3 + 1 = ((\sqrt{2})^2 - 2)((\sqrt{2})^3 - 6(\sqrt{2})) + (1 + 12(\sqrt{2})) = (0)((\sqrt{2})^3 - 6(\sqrt{2})) + (1 + 12(\sqrt{2}) = 1 + 12\sqrt{2}\)
2 Coordinates, Lines, Linear and Quadratic Polynomials, Functions

2.1 September 23

1. Find all values of \( x \) such that \( |2x - 3| = 7 \).
   \( \text{Ans: } 2x - 3 = 7 \) or \( -(2x - 3) = 7 \), \( x = 5 \) or \( x = -2 \)

2. The sum of two numbers is 13 and twice one of them plus three times the other is 12. The numbers are: _______. Show your work.
   \( \text{Ans: } 27, -14 \)

3. Find the value of \( w \) so that the determinant of the matrix \( \begin{vmatrix} -6 & w \\ 3 & 5 \end{vmatrix} \) is 12. Show your work.
   \( \text{Ans: } 14 \)

4. A triangle and a 5 by 8 rectangle are cut out of a square of unknown side \( x \), leaving the shaded region indicated in the diagram.

   ![Diagram showing a square with a triangle and a rectangle cut out, leaving a shaded region.]

   Express the area \( A \) of the region as a polynomial in \( x \). _______

   \( \text{Ans: } A(x) = \frac{1}{2}x^2 - 40 \)

5. A right triangle with sides of length 3 and 7 is enlarged with the side of length 7 increased by an amount \( 5w \) and the side of length 3 increased by \( w \). Express the area of the resulting triangle in the form \( A(w) = aw^2 + bw + c \).

   ![Diagram of an enlarged right triangle with sides labeled.]

   \( \text{Ans: } \frac{1}{2}(7 + 5w)(3 + w) \)
2.2 September 25

1. Tom and Jerry live on the same straight road. Each of them uses his own set of coordinates for points on the road. Tom uses \( x \) for his coordinates and Jerry uses \( z \). Tom says that the \( x \)-coordinate of his house is 3 and that the \( x \) coordinate of Jerry’s house is 7. Jerry says that the \( z \) coordinate of his house is 0 and that the \( z \) coordinate of Tom’s house is 12.

![Diagram showing Tom, Sally, and Jerry's houses on a road.]

Tom says that Sally’s house has \( x \)-coordinate 4. What coordinate does Jerry assign to Sally’s house? \( \alpha \) [Note that the diagram is not necessarily to scale.]

Ans: \( z = \alpha x + \beta \),

\[
\begin{array}{ccc}
\text{Tom} & x & 3 \\
\text{Jerry} & z & 12 \\
\text{Sally} & & 0 \\
\end{array}
\]

\[
12 = 3\alpha + \beta, 0 = 7\alpha + \beta, \alpha = -3, \beta = 21,
\]

\[
z = (-3)(4) + 21 = 9
\]

2. The line \( L \) has two sets of coordinates “\( x \)” and “\( x' \)” where the \( x \) and \( x' \) coordinates of a point are related by the formula \( x' = ax + b \). The point \( A \) has \( x \) coordinate 2 while its \( x' \) coordinate is -3.

The point \( B \) has \( x \) coordinate 5 while its \( x' \) coordinate is 7. The formula for translating from \( x \) to \( x' \) coordinates is \( x' = \) \( \) _______. The point \( C \) whose \( x' \) coordinate is 1 has \( x \)-coordinate _______.

Ans: \( x' = \frac{10}{3}x - \frac{29}{3}, \frac{3}{10}(1 + \frac{29}{3}) \)

3. The line \( M \) has two sets of coordinates “\( z \)” and “\( w \)” where the \( z \) and \( w \) coordinates of a point are related by the formula \( w = az + b \). The point \( P \) has \( z \) coordinate 0 and \( w \) coordinate 5.

The point \( Q \) has \( z \) coordinate 1 and \( w \) coordinate is 3. The formula for translating from \( z \) to \( w \) coordinates is \( z = \) \( \) _______. The point \( S \) whose \( w \) coordinate is -1 has \( z \) coordinate _______.

Ans: \( w = -2z + 5, 3 \)

2.3 September 28

1. Find all the points \( P(x, y) \) on the \( y \)-axis which are \( \sqrt{29} \) units away from the point \( A(-5, -8) \).

The \( y \)-coordinates of the points are \( \) _______.

Ans: \([ -10, -6 ]\)
2. Each point in the plane has coordinates \((x, y)\) relative to the coordinate axes on the left and \((z, w)\) relative to the axes on the right. The formulas for changing from \((x, y)\) coordinates to \((z, w)\) coordinates are \(z = ay + 4\) and \(w = -x + d\).

The point \(P\) has \(x - y\) coordinates \((2, 3)\) and \(z - w\) coordinates \((-1, 5)\).

(a) The origin, \(O_{xy}\) in the \(x - y\) coordinate system has \(z - w\) coordinates with \(z = \) _____ and \(w = \) _____.

(b) The point \(Q\) that has \(z - w\) coordinates \((3, 7)\) has \(x - y\) coordinates with \(x = \) _____ and \(y = \) _____.

\[
\begin{array}{c|c|c}
   & x - y & z - w \\
\hline
\text{O}_{xy} & (0, 0) & \text{ } \\
Q & (3, 7) & \text{ } \\
P & (2, 3) & (-1, 5) \\
\end{array}
\]

Ans: \(x = \), \(y = \)

Observe that the answer to (b) as \((7, -8)\) while the correct one is \((7, 8)\).

3. Suppose we have \((x, y)\) and \((z, w)\) coordinates for points in the plane. If the formulas \(\{z = -x + 7, w = y + 8\}\) define the transformation from \((x, y)\) to \((z, w)\) coordinates, then the formulas defining the reverse transformation from \((z, w)\) to \((x, y)\) coordinates are:

\[x = \text{_____}, y = \text{______} .\]

(b) If \(O_{xy}\) is the origin in the \(x - y\) coordinate system then what are the the \(z - w\) coordinates for \(O_{xy}\)? Ans: \(x = -z + 7, y = w - 8\) (b) \(z = -(0) + 7, w = 0 + 8\) so the coordinates are \((7, 8)\).
2.4 September 30

1. The transformation \( T \) maps the plane onto itself by multiplication by a complex number. That is, there is a complex number \( C = a + ib \) such that any point \( P(x, y) \) has image \( T(P) \) as the point corresponding to the complex number \( CP \). That is \( T(P) = (ax - by, bx + ay) \) since \((a + bi)(x + iy) = (ax - by) + (ay + bx)i\).

For a particular complex number \( C \) the transformation \( T \) takes the smaller triangle in the diagram to the larger one. The point \( A(-1, 1.5) \) (the upper left vertex) on the smaller triangle is taken to the point \( T(A) = (-\frac{11}{2}, 5) \) on the larger triangle.

(a) The complex number \( C \) must be equal to _________

Ans: \( 4 + i \)

(b) The small triangle is rotated by _________ degrees counterclockwise and expanded by a factor of _________.

Ans: \( \arctan(1/4) = 14.03 \) deg, \( \sqrt{17} \)

2. A transformation \( T \) maps the plane to the plane by multiplication by a complex number. That is we view the point \( P(x, y) \) as the complex number \( P = x + iy \) and there is a complex number \( C = a + bi \) such that \( T(P) = CP \).

The transformation takes the point \( A \) whose coordinates are \((5, 2)\) to the point \( T(A) \) whose coordinates are \((14, 23)\)

a.) The complex number \( C \) is _________

b.) The effect of \( T \) is to rotate each point \( P \) by ________ degrees counterclockwise about the origin and to stretch its direction from the origin by a factor of _________.

Ans: (a) \( \frac{14+23i}{5+2i} = 4 + 3i \), (b) \( \arctan(3/4) = 36.86989 \) deg, 5
2.5 October 2

1. The line \( L \) is described in point-slope form by the equation \( y - 3 = 5(x - 2) \) and the line \( M \) is described parametrically by the equations \( \{x = 3 + t, y = 8 + 2t\} \).

The point of intersection of the two lines is (____, ____).

Ans: \((8 + 2t) - 3 = 5(3 + t) - 3, t = 0, pt = (3, 8)\).

2. If \( A = (-1, 3) \) and \( B = (5, 7) \), the point \( P \), on the line segment \( \overline{AB} \) whose distance from \( B \) is twice its distance from \( A \) is

Ans: The line segment is \( A + t(B - A) \) for \( 0 \leq t \leq 1 \) with 0 corresponding to \( A \) and 1 to \( B \). On the interval \([0, 1]\) with \( A = 0 \) and \( B = 1 \), \( t = \frac{1}{3} \) so \( P = A + \frac{1}{3}(B - A) = (-1, 3) + \frac{1}{3}(6, 4) = (1, \frac{13}{3}) \),

3. Determine parametric equations \( x(t) \) and \( y(t) \) for the line through the points \( A(4, 7) \) and \( B(-5, 9) \) such that the point \( A \) corresponds to \( t = 0 \) and \( B \) corresponds to \( t = 1 \).

Ans: \( x(t) = 4 + t \ast (-9), y(t) = 7 + t \ast (2) \)

4. The line \( L \) is given parametrically by \( x(t) = 3t - 5, y(t) = -2t + 7 \). The cartesian slope-intercept \((y = mx + b)\) equation for \( L \) is

Ans: \( y = -\frac{2}{3}x + \frac{11}{3} \)

5. Suppose \( A = (1, 3), B = (4, 9), \) and \( C = (2, 7) \). One set of parametric equations for the line through \( C \) which is parallel to the line through \( A \) and \( B \) is

Ans: One set is \( C + t(B - A) = (2, 7) + t(3, 6) = (3t + 2, 6t + 7) \), \( x(t) = 3 + 2t, y(t) = 6t + 7 \)

6. The line \( L \) through the points \( A(1,13) \) and \( B(7,5) \) is given parametrically by the equations \( x(t) = \:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\\)\), y(t) = \:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\\)\)\)\)\)\).

Ans: The answer is of course not unique. For the parameterization \( p(t) = A + t(B - A) \) we have \( x(t) = 1 + 6t, y(t) = 13 - 8t, (a) -1, (b) \frac{5}{6}, (1 + \frac{5}{6} \ast 6, 13 - \frac{5}{6} \ast 8) \)

2.6 October 5

7. What is the point of intersection of the line \( L \) which has parametric equations \( x(t) = 2t + 1, y(t) = -3t + 16 \) and the line with equation \( y = x \)?

Ans: \((7, 7)\)
8. The equation of the line which is perpendicular to the line with equation \( y = -\frac{1}{3}x - 7 \) and which passes through the point \((2, 5)\) is \( y = \underline{3}x + \underline{1} \)

**Ans:** \( y = 3x - 1 \)

9. A set of parametric equations for the line with cartesian equation \( y = 2x + 3 \) is

**Ans** \( x(t) = t, y(t) = 2t + 3 \) is one set.

10. A set of parametric equations for the vertical line with cartesian equation \( x = 12 \) is

**Ans** \( x(t) = 12, y(t) = t \) is one set.

11. For which value of \( y \) will the the lines connecting \((-2, 3)\) and \((5, y)\) meet at the origin in a 90 degree angle?

\[ y = \underline{3}y - 5, 3y = 10, y = \frac{10}{3} \]

**Ans:** \( y = \frac{3}{2} = -\frac{5-0}{y-0}, 3y = 10, y = \frac{10}{3} \)

12. What are the coordinates of the point on the line \( y = x + 1 \) which is exactly the same distance from the point \((2, 3)\) as it is from the \((0, 0)\)

\( \underline{(\frac{7}{10}, \frac{17}{10})} \)

**Ans:** \((\frac{7}{10}, \frac{17}{10})\)

2.7 October 7

13. What is the point of intersection of the line \( L \) which has parametric equations \( x(t) = 2t, y(t) = t - 1 \) and the line \( M \) which has parametric equations \( x(s) = -s + 1, y(s) = 5s \)?

**Ans** \( \{2t = -s + 1, t - 1 = 5s\} \), yields \( s = -\frac{1}{11}, t = \frac{6}{11} \). Use either value, e.g., \( x = 2\left(\frac{6}{11}\right), y = \frac{6}{11} - 1 \)

14. If \( A \) and \( B \) are points in the plane and \( t \) is a parameter then \( P(t) = tA + (1 - t)B \) is the (linear) parametric form of the line \( L \) through \( A \) and \( B \) such that \( P(0) = B \) and \( P(1) = A \). If
$M = (B - A)$ then one also has $P(t) = tM + B$ In which $M$ may be viewed as a “slope”. If $M = (a, b)$, let $M^\perp = (b, -a)$. Then if $C$ is any point $Q(t) = C + tM^\perp$ is a parametric form of the line perpendicular to $L$ and passing through $C$.

If $A = (4, 6), B = (9, -4)$, and $C(5, -9)$, find the above parametric form, $Q$, of the line perpendicular to the line through $A$ and $B$ and passing through $C$.

**Ans:** $Q(t) = (-10t + 5, -5t - 9)$

15. If $A = (2, -5), B = (5, -1)$, find parametric equations for the perpendicular bisector of the line segment $\overline{AB}$

**Ans:** midpt $= \frac{1}{2}(A + B) = (\frac{7}{2}, -3)$, direction (generalized slope) $= B - A = (3, 4)$, normal direction $= (B - A)^\perp = (-4, 3)$, $(x(t), y(t)) = (\frac{7}{2}, -3) + t(-4, 3))$

### 2.8 October 9

1. $S(t) = (45, 40, 60) + \frac{1}{7}(25, 30, 10)$ so $x(t) = 45 + \frac{1}{7}(25), y(t) = 40 + \frac{1}{7}(30), z(t) = 60 + \frac{1}{7}(10)$

2. Find the point of intersection of the line $L$ which passes through points $A(-5, 8)$ and $B(1, 2)$ and the line $M$ which passes through $C(-3, 2)$ and $D(4, 1)$.

**Ans:** $L$ has parametrization $P(t) = A + t(B - A)$ and $M Q(s) = C + s(D - C)$. The point of intersection determined by the values of $t$ and $s$ for which $P(t) = Q(s)$ or $(-5, 8) + t((1, 2) - (-5, 8)) = (-3, 2) + s((4, 1) - (-3, 2))$ (6t - 5, -6t + 8) = (7s - 3, -s + 2) this gives 6t - 6 = 7s - 3 -6t + 8 = -s + 2 which have $s = \frac{2}{3}$, $s = \frac{10}{9}$ as the unique solution.

It checks that $P(\frac{10}{9}) = Q(\frac{2}{3}) = (\frac{5}{3}, \frac{4}{3})$

3. A laser beam passes through the points $A(-5, 8)$ and $B(1, 2)$ and is reflected off of the line through $C(-3, 2)$ and $D(4, 1)$. Find parametric equations for the line $R$ along which the beam travels after it is reflected.

**Ans:** From the previous problem the point of intersection of $L$ and $M$ is $(\frac{5}{3}, \frac{4}{3})$. The direction of $L$ is $B - A = (6, -6)$ and that of $M$ is $D - C = (7, -1)$. We write $(B - A) = \alpha(D - C) + \beta(D - C)^\perp (6, -6) = \alpha (7, -1) + \beta(1, 7)$ which gives equations

$6 = \alpha 7 + \beta$

$-6 = \alpha (-1) + \beta (7)$

Which has solution $\alpha = \frac{24}{25}$, $\beta = -\frac{18}{25}$

The direction for the reflected line is $\alpha(D - C) - \beta(D - C)^\perp = (\frac{179}{25}, \frac{78}{25})$

So the parameterization of the reflected line $R$ is

$R(w) = (\frac{5}{3}, \frac{4}{3}) + w(\frac{186}{25}, \frac{102}{25})$
3 October 12

1. The answer for this problem had a sign error which has been corrected.

A phone cable runs between two poles. The base of the shorter (red) pole is located at the point (20, 10) and that of the taller (black) pole is at (45, 40) (coordinates are in feet). The shorter pole is 50 feet high and the other is 60 feet high. (The diagram is not to scale.)

Beginning at time \( t = 0 \) seconds a squirrel runs at a constant velocity along the cable from the top of the higher pole to the top of the lower pole in 7 seconds. Parametric equations for the position (in space) of the squirrel at time \( t \) seconds for \( t \) in the interval \([0, 7]\) are \( x(t) = \underline{\phantom{1}}, y(t) = \underline{\phantom{1}}, z(t) = \underline{\phantom{1}} \).

Ans: \( S(t) = (x(t), y(t), z(t)) = (45, 40, 60) + \frac{t}{7}((20, 10, 50) - (45, 40, 60)) \) moves from the top of the taller pole to top of the shorter as \( t \) goes from 0 to 7.

2. There was also a sign error in this problem. In fact the two lines don’t meet.

Let \( A = (1, 2, 3), B = (0, 0, 1), C = (-4, -2, 5), \) and \( D = (-1, 1, 0) \). Calculate the point of intersection of the line through \( A \) and \( B \) with the line through \( C \) and \( D \).

Ans: If we write \( P = A + t(B - A) \) and \( Q = C + s(D - C) \) then we get \( P = (-t + 1, -2t + 2, -2t + 3), Q = (3s - 4, 3s - 2, -5s + 5) \). In order for these to meet then there must be \( s \) and \( t \) such that \(-t + 1 = 3s - 4, -2t + 2 = 3s - 2, -2t + 3 = -5s + 5\). When we solve the first pair we get \( s = 2, t = -1 \) but when we solve the second and third we get \( s = \frac{3}{4}, t = \frac{7}{8} \) which means there is no common solution so the lines do not meet.

4 October 14

1. Suppose \( f(x) = x^2 + 3x + 6 \). Determine the value of \( s \) such that \( f(u + s) = u^2 + C \) where \( C \) is a constant. In other words, in the expansion of \( f(u + s) \), the coefficient of \( u \) is zero. \( s = \underline{\phantom{1}} \) Justify your answer.

With your value of \( s \) the value of \( C \) is \( \underline{\phantom{1}} \). Justify your answer.

Ans: \( -\frac{3}{2}, \frac{15}{4} \)

2. Each of the graph sketches below is that of a function of the form \( f : \mathbb{R} \to \mathbb{R} \) where \( f(x) = ax + b \). Complete the table below by entering "+", "-", "0", or "∞" to indicate the signs (or non-existence) of the coefficients or expressions for each of the graphs A-J.
Each of the graph sketches below is that of a function of the form \( f : \mathbb{R} \to \mathbb{R} \) where \( f(x) = ax^2 + bx + c \). Complete the table below by entering “+”, “−”, “0”, or “∞” to indicate the signs (or non-existence) of the coefficients or expressions for each of the graphs \( A \rightarrow J \).

<table>
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<tr>
<th></th>
<th>( a )</th>
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<th>( c )</th>
<th>( \frac{-b}{2a} )</th>
<th>( b^2 - 4ac )</th>
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3. Each of the graph sketches below is that of a function of the form \( f : \mathbb{R} \to \mathbb{R} \) where \( f(x) = ax^2 + bx + c \). Complete the table below by entering “+”, “−”, “0”, or “∞” to indicate the signs (or non-existence) of the coefficients or expressions for each of the graphs \( A \rightarrow J \).

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5 October 16

1. **Definition:** The function \( f : D \to T \) is **one-to-one (injective)** if \( \cdots \)
   
   **Ans:** \( f(a) = f(b) \) implies that \( a = b \).
   
   Alternately, if \( a \neq b \) then \( f(a) \neq f(b) \).

2. **Definition:** The function \( f : D \to T \) is **onto (surjective)** if \( \cdots \)
   
   **Ans:** for every \( t \in T \) there is \( d \in D \) such that \( f(d) = t \).

3. **Definition:** \( f : S \to \mathbb{R} \) has an **absolute maximum value** at \( x = t \) if \( F(x) \leq F(t) \) for every \( x \in S \).

4. **Definition:** \( f : S \to \mathbb{R} \) has an **absolute minimum value** at \( x = t \) if \( F(t) \leq F(x) \) for every \( x \in S \).

5. **Definition:** \( f : S \to \mathbb{R} \) has an **absolute extremum value** at \( x = t \) if \( f \) has either an absolute maximum or an absolute minimum at \( x = t \).

6. The absolute extremum of the quadratic function \( f(x) = ax^2 + bx + c \) occurs at \( \square \). The function has an \( \square \) of \( \square \) there if \( a > 0 \) and an \( \square \) of \( \square \) there if \( a < 0 \).

   **Ans:** \( \frac{-b}{2a} \), absolute minimum, \( f(\frac{-b}{2a}) \), absolute maximum, \( f(\frac{-b}{2a}) \).
7. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x - 7$. Show that $f$ is both injective (one to one) and onto (surjective).

**Ans:** Suppose $f(r) = f(s)$ then $3r - 7 = 3s - 7$ so $3(r - s) = 0$ and $r = s$. Thus $f$ is injective.
For “onto” we must take any $t$ in the target (all of $\mathbb{R}$) and show that there is $d$ in the domain (also all of $\mathbb{R}$) such that $f(d) = t$. So we need to solve $f(d) = t$ or $t = 3d - 7$. Solving for $d$ we have $d = \frac{t+7}{3}$ so there is $d$ such that $f(d) = t$.

8. If $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2 + 5x + 3$, show that $f$ is neither one to one nor onto.

**Ans:** “Completing the square” we set $x = u - \frac{5}{2}$ have that $f(x) = u^2 - (\frac{5}{2})^2 + 3$.
For any value of $u$ that is not zero, $u \neq -u$ but $f(u) = f(-u)$ so $f$ is not one to one. Moreover the smallest value of $f$ is $(\frac{5}{2})^2 + 3$ when $u = 0$ ($x = -\frac{5}{2}$) so if $t < (\frac{5}{2})^2 + 3$ there is no value of $d$ such that $f(d) = t$.

9. Suppose $f : [1, 2] \to [1, 5]$ is defined by $f(x) = -4x + 9$. Show that $f$ has an inverse amd calculate $f^{-1}(x)$ for any $x$.

(b) Verify that for any $x$ $f^{-1}(f(x)) = x$ and for any $t$ $f(f^{-1}(t)) = t$.

**Solution:** For linear functions $f([a, b]) = [f(a), f(b)]$ so $f([1, 2]) = [f(1), f(2)] = [5, 1]$ which we write as $[1, 5]$. Thus the image of $f$ is its target and $f$ is surjective. To see that $f$ is injective we set $f(a) = f(b), -4a + 9 = -4b + 9, -4a = -4b, a = b$ so $f$ is injective. To calculate the rule for the inverse we set $x = f(y)$

$x = -4y + 9$

$y = \frac{x-9}{4}$

$f^{-1}(x) = \frac{x-9}{4}$ For part (b) we calculate $f^{-1}(f(x)) = \frac{(-4x-9)+9}{-4} = x$

$f(f^{-1}(t)) = -4(\frac{t-9}{4}) + 9 = t$

6 October 19

1. The coordinates of the point $A$ are $(5, 4)$ and those of the point $B$ are $(9, 3)$. What are the coordinates of the point which is on the $x$-axis and such that the triangle $PAB$ is a right triangle with the right angle at the point $A$?

**Ans:** A parametrization of the line $L$ through $A$ and $B$ is $P(t) = A + t(B - A) = ((5, 4) + t((9, 3) - (5, 4))) = (5, 4) + t(4, -1)$. The perpendicular direction to $L$ is $(4, -1)^\perp = (1, 4)$ and the line through $A$ with this direction is $Q(s) = (5, 4) + s(1, 4) = (5 + s, 4 + 4s) = (x(s), y(s))$. This line meets the $x$ axis when $y(s) = 0$ or $s = -1$ so the point $C$ is $(4, 0)$. 

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Alternately one can set \( C = (t, 0) \) then the slope of \( AC = \frac{4-0}{5-t} \) and the slope of \( AB = \frac{3-4}{9-5} = \frac{-1}{4} \). We want \( \overline{AB} \) and \( \overline{AC} \) perpendicular to their slopes must be negative reciprocals so we must have \(-\frac{1}{4} = \frac{4-0}{5-t} \) so
\[
4 = \frac{4}{5-t} \\
1 = \frac{1}{5-t} \\
5 - t = 1 \text{ or } t = 4
\]

2. If \( f : D \rightarrow T \) is function then \( D \) is called the \underline{domain}, and \( T \) the \underline{target} of \( f \). The range of \( f \) is \( \{ f(x) \mid x \in D \} \).

\[ \text{Ans: domain, target, } \{ f(x) \mid x \in D \} \]

3. If \( f(x) = ax^2 + bx + c \) with \( a \neq 0 \) then the \underline{discriminant} of \( f \) is \underline{\( b^2 - 4ac \)}, the \( y \)-intercept is \underline{\( c \)}, the \( x \)-coordinate of the \underline{vertex} is \underline{\( -\frac{b}{2a} \)} and the \( y \)-coordinate of the \underline{vertex} is \underline{\( f(-\frac{b}{2a}) \)}

\[ \text{Ans: } b^2 - 4ac, \ c, \ -\frac{b}{2a}, \ f(-\frac{b}{2a}) \]

### 7 Additional Review Problems for Exam II

1. The points of intersection of the line given by the equation \( y = x + 3 \) and the circle which satisfies the equation \( x^2 + y^2 - 2x - 79 = 0 \) are \( (\underline{\phantom{0}}, \underline{\phantom{0}}) \) and \( (\underline{\phantom{0}}, \underline{\phantom{0}}) \).

\[ \text{Ans: } x^2 + (x+3)^2 - 2x - 79 = 0, \ 2(x-7)(x+5) = 0, \ x = 7 \text{ gives pt } (7, 10), \ x = -5 \text{ gives } (-5, -3) \]

2. The line \( L \) has parametric equations \( x(t) = -5t + 1, y(t) = 2t + 11 \).

(a) Find parametric equations \( x(t), y(t) \) for the line parallel to \( L \) which passes through the origin \( O(0,0) \).

(b) Find parametric equations for the line \( M \) which is perpendicular to \( L \) and passes through the origin \( Q(0,0) \)

\[ \text{Ans: One set of solutions is: (a) } x(t) = -5t, y(t) = -2t, (b) x(t) = 2t, y(t) = -5t \]

3. The line \( L \) is given by the parametric equations \( \{ x = 6t - 9, y = -3t + 7 \} \). Let \( A \) be the point corresponding to the value \( t = -1 \) and \( B \) the point corresponding to the value \( t = 1 \). What is the equation of the line which is the perpendicular bisector of the line segment \( AB \)?

\[ \text{Ans: } \]

4. Let \( L \) be the line with equation \( 8x + 5y + 3 = 0 \).

(a) A line parallel to \( L \) and passing through the point \( M(-6,6) \) is given by \underline{\phantom{0}}x + \underline{\phantom{0}}y + \underline{\phantom{0}} = 0 \)

\[ \text{(a) A line parallel to } L \text{ and passing through the point } M(-6,6) \text{ is given by } x + y + 21 = 0 \]
(b) A line perpendicular to \( L \) and passing through the point \( M(-6, 6) \) is given by \( \_ \_ \_ \_ \_ \_ x + \_ \_ \_ \_ \_ \_ y + \_ \_ \_ \_ \_ \_ = 0 \)

\textbf{Ans:} (a) \( 8x + 5y - 78 = 0 \), (b) \( -5x + 8y + 18 = 0 \)

5. Consider the parametric line: \( x = 3t + 1, y = t + 1 \). Determine the slope-intercept equation for the same line.
\textbf{Ans:} \( y = \frac{1}{3}x + \frac{2}{3} \)

6. Consider two parametric lines:
\textit{Line 1}: \( x = 2t + 2, \ y = -3t + 1 \)
\textit{Line 2}: \( x = 3s + 3, \ y = -s + 1 \)
Let \( P(x, y) \) be their common point.
The \( x \)-coordinate of \( P \) is \( \_ \_ \_ \_ \_ \_ \) and its \( y \)-coordinate is \( \_ \_ \_ \_ \_ \_ \).

\textbf{Ans:} \( \frac{12}{7}, \frac{4}{7} \)

7. Each point in the plane has coordinates \((x, y)\) relative to the coordinate axes on the left and \((z, w)\) relative to the axes on the right. The formulas for changing from \((x, y)\) coordinates to \((z, w)\) coordinates are \( z = ay + 4 \) and \( w = -x + d \).

The point \( P \) has \( x - y \) coordinates \((2, 3)\) and \( z - w \) coordinates \((-1, 5)\).

(a) The origin, \( O_{xy} \) in the \( x - y \) coordinate system has \( z - w \) coordinates with \( z = \_ \_ \_ \_ \_ \_ \) and \( w = \_ \_ \_ \_ \_ \_ \).

(b) The point \( Q \) that has \( z - w \) coordinates \((3, 7)\) has \( x - y \) coordinates with \( x = \_ \_ \_ \_ \_ \_ \) and \( y = \_ \_ \_ \_ \_ \_ \).

\begin{tabular}{c|cc}
\hline
\( x - y \) & \( z - w \) \\
\hline
\( O_{xy} \) & \((0,0)\) & \((3,7)\) \\
\( Q \) & \((2,3)\) & \((-1,5)\) \\
\( P \) & \((2,3)\) & \((-1,5)\) \\
\hline
\end{tabular}

\textbf{Ans:} \( z = ay + 4, \) from \( P: \) \( -1 = 2a + 4, \ a = \frac{5}{2} \)
\( w = -x + d, \) from \( P: \) \( 5 = -2 + d, \ d = 7 \)
so the conversion is \( z = \frac{-5}{3}y + 4, \ w = -x + 7 \)

\( O_{xy}: \ z = \frac{-5}{2}0 + 4 \) so \( z = 4, \ 0 = -x + 7 \) so \( O_{xy} = (4, 7) \) in \( x - w \)

\( Q : \) has \( z - w \) coords \( (3, 7) \), \( 3 = \frac{-5}{3}y + 4 \) so \( y = \frac{3}{5}, \ w = -x + d \) so \( 7 = -x + 7 \) so \( x = 0 \) and the \( x - y \) coords of \( Q \) are \( (0, \frac{3}{5}) \)

8. Find the point of intersection of line \( M \) with point slope equation \( y = -3x + 4 \) and the line \( L \) that passes through \( A(-1, 2) \) and \( B(6, -5) \).

**Ans:** Parametric equations for \( L \) are \( P(t) = A + t(B - A) = (7t - 1, -7t + 2) = (x(t), y(t)) \). The point \( P(t) \) is on \( M \) if and only if \( y(t) = -3x(t) + 4 \)

\(-7t + 2 = -3(7t - 1) + 4\)

\[-14t = 5\]

\[t = \frac{5}{14}\]

\[P\left(\frac{5}{14}\right) = (7\left(\frac{5}{14}\right) - 1, -7\left(\frac{5}{14}\right) + 2) = (\frac{3}{2}, -\frac{1}{2})\]

check: \(-\frac{1}{2} = -3(\frac{3}{2}) + 4\)

8 **October 21**

There was no class on October 21.

9 **October 23**

1. A laser beam passes through the points \( A(-5, 2) \) and \( B(-1, -1) \). It meets the line \( L \) through \( O(0, 0) \) and \( C(1, 1) \) at the point \( B \) and is reflected. Find parametric equations for the line \( R \) along which the beam travels after it is reflected.

**Ans:** Let \( M \) be the line through \( A \) and \( B \). Then a parametric equation for \( M \) is \( M(t) = A + t(B - A) \) so the direction of \( M \) is \( B - A = (4, -3) \).

The direction of \( L \) is \( (C - O) = (1, 1) \)

We resolve the direction of \( M \) into its component that is parallel to \( L \) and its component that is parallel to \( L \).

Write \( (4, -3) = \alpha(C - O) + \beta(C - O)^\perp \)

\[(4, -3) = \alpha(1, 1) + \beta(-1, 1) = (\alpha - \beta, \alpha + \beta)\]

\[\alpha - \beta = 4\]

\[\alpha + \beta = -3\]

\[\alpha = \frac{1}{2}, \ \beta = -\frac{7}{2}\]

The direction of \( R \) is \( \alpha(C - O) - \beta(C - O)^\perp \)

\[\frac{1}{2}(1, 1) - \frac{7}{2}(-1, 1) = (-3, 4)\]

So a \( R(s) = (-1, -1) + s(-3, 4) \)

10 **October 26**

1. What is the equation of the circle which has the line segment connecting \((4, 1)\) and \((-2, -7)\) as a diameter?
\[(x - \underline{______})^2 + (y - \underline{______})^2 = \underline{______}\]

Ans:
\[(x - 4)(x + 2) + (y - 1)(y + 7) = 0 , (x - 1)^2 + (y - 3)^2 = 5^2\]

2. The circle \(C_1\) is given by the equation \((x - 2)^2 + (y + 1)^2 = 5\). What are the center and radius of \(C_1\)?
Center = \underline{______} , radius = \underline{______}

Ans: Center = \((2, -1)\) , radius = \(\sqrt{5}\).

3. The circle \(C_2\) is given by the equation \(x^2 + 2x - 20 + y^2 - 4y = 0\). What are the center and radius of \(C_1\)?
Center = \underline{______} , radius = \underline{______}

Ans: Center = \((-1, 2)\) , radius = 5.

4. The line connecting the points of intersection of the circles with equations \((x - 2)^2 + (y + 1)^2 = 5\) and \((x + 1)^2 + (y - 2)^2 = 25\) is \(\underline{______}x + \underline{______}y + \underline{______} = 0\)

Ans: \(((x - 2)^2 + (y + 1)^2 = 5) - ((x + 1)^2 + (y - 2)^2 = 25) = -6x + 6y + 20 = 0\)

11 November 4

1. 25 degrees = \underline{______} radians.

Ans: \(25 \text{ deg} \times \frac{\pi \text{ rad}}{180 \text{ deg}}\)

2. 2.5 radians = \underline{______} degrees.

Ans: \(2.5 \text{ rad} \times \frac{180 \text{ deg}}{\pi \text{ rad}}\)
3. At 5:45 the angle made by the hands of a clock measures \[ \frac{13\pi}{24} \text{ rad} \].

Ans: \[ \frac{13\pi}{24} \]

4. The vertex of the pie-slice region below is at the center of the circle. The radius of the circle is 10 feet and the length of the circular arc that forms part of the boundary of the shaded region is 12 feet long.

The radian measure of the shaded angle which has its vertex at the center of the circle is \[ \frac{12}{10} \text{ rad} \].

Ans: \[ \frac{12}{10} \]

5. The area of the pie-slice region below is 50 square feet and the area of the entire circle is 500 square feet. The radian measure of the shaded angle which has its vertex at the center of the circle is \[ \frac{2\pi}{10} \text{ rad} \].

Ans: \[ \frac{2\pi}{10} \]

\[ \text{\textbf{Ans:}} \frac{2\pi}{10} \]
1. Each of the labeled points on the above unit circle is the locator point $P(\theta)$ of a number $\theta$ which is between $0$ and $2\pi$ and of the form $n\frac{\pi}{12}$, $n$ an integer. Each $\theta$ is the radian measure of the angle $\angle AZX$ where $X$ is any of the of the letters $A...P$ and the coordinates of the point $P(\theta)$ are the cos and sin of the geometric angle $\angle AZX$ and are also the values of the functions $\cos(x)$ and $\sin(x)$ for $x = \theta$.

Complete the following table. All entries must be exact.

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<th>$\theta$</th>
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13  November 11

1. Complete the following to express $\sin(a + b)$ and $\cos(a + b)$ in terms of $\sin(a), \cos(a), \sin(b), \cos(b)$.

(i) $\sin(a + b) = \underline{\text{______}}$.

\textbf{Ans:} $\sin(a) \cos(b) + \sin(b) \cos(a)$

(ii) $\cos(a + b) = \underline{\text{______}}$.

\textbf{Ans:} $\cos(a) \cos(b) - \sin(a) \sin(b)$

2. \texttt{scriptsize} The answer to part (a) was edited Nov 10 to correct a typo. For each of the clocks shown below, give the radian measure of the angle through which the minute hand must rotate counterclockwise to overlap the hour hand.

\begin{enumerate}
\item[(a)] \\
\item[(b)] \\
\item[(c)] \\
\item[(d)]
\end{enumerate}

\textbf{Ans:} (a) $\frac{3}{4} \pi$, (b) $\frac{3}{2} \pi$ (c) $\frac{67}{72} \pi$ (d) $\frac{\pi}{24}$

3. On an analog clock the hands of are together for the first time after midnight at ________ .

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{clock}
\end{figure}

At this time the hands of the clock make an angle of ________ radians with the ray they define at midnight.

\textbf{Ans:} $\frac{60}{11}$ minutes after midnight., $\frac{2\pi}{11}$ radians.
4. In the triangle \(ABC\) (as represented in the diagram) the angle \(B\) is .439 radians, the side \(BC\) has length 20 and the side \(AB\) has length 21.

\[\text{Ans: } \sqrt{20^2 + 21^2 - 2 \times 20 \times 21 \cos(.439)}\]

14 November 13

1. Given the information in the table, what is the length of \(c\)?

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(t)</th>
<th>(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 cm</td>
<td>8 cm</td>
<td></td>
<td>5 cm</td>
<td>15 deg</td>
<td></td>
</tr>
</tbody>
</table>

\[\text{Ans: Use } \tan(s + t) = \frac{b+c}{a}\text{ to get } c = 1.323394458\text{ cm}\]

2. If \(\angle B\) measures \(\frac{\pi}{12}\) and \(a = c = 10\) then what is \(b\)?
15 November 16

(a) With respect to the diagram (which is not to scale), \( \triangle ABC \) is a right triangle with the right angle at \( B \), the length of the line \( BC \) is 17 inches, the angle \( \angle BAC \) is \( \frac{\pi}{4} \), and the angle \( \angle ACM \) is \( \frac{\pi}{6} \). What is the length of \( AM \)?

Ans: \( 1 = \tan\left(\frac{\pi}{4}\right) = \frac{17}{AB} \) so \( AB = 17 \).
\( \angle BAC \) is \( \frac{\pi}{2} - \frac{\pi}{4} \) so \( \angle ACB \) is also \( \frac{\pi}{4} \).
Angle \( \angle MCB \) is \( \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \).
Therefore \( \frac{BM}{17} = \tan\left(\frac{\pi}{12}\right) \) so \( AM = 17 - 17\tan\left(\frac{\pi}{12}\right) = 12.44486373 \) inches.

(b) What is the equation of the circle which has a diameter with end points \( A(3, 5) \) and \( (8, -7) \)

Ans: \( (x - 3)(x - 8) + (y - 5)(y + 7) = 0 \)

(c) If \( \angle A \) measures 100 degrees, angle \( B \) 20 degrees, and side \( c \) is 15 inches? What are the lengths of \( a \) and \( b \)?

Ans: \( \frac{\sin(C)}{c} = \frac{\sin(A)}{a} \)
\( \frac{\sin(60^\circ)}{15} = \frac{\sin(100^\circ)}{a} \)
\( a = 17.05737064 \)

(d) The segment \( \overline{AB} \) is 5 inches and the segment \( \overline{BC} \) is 4 inches. If angle \( \angle CAD \) measures 30 degrees then what is the length of the segment \( \overline{CD} \)?
Ans: Use \( \tan \left( \frac{7}{6} + \angle BAC \right) = \frac{4 + x}{5} \) 
8.797803 inches

(e) The angle \( \angle CAB \) in the diagram measures 75 degrees and the circle is centered at \( A \) and has a radius of 7 inches.

\begin{align*}
\text{a) What is the area of the red triangle?} \\
\text{Ans: } 75 \text{ deg} \times \left( \frac{\pi \text{ rad}}{180 \text{ deg}} \right) = \frac{5\pi}{12} \text{ rad} \\
\text{Area = area of two half-triangles} = 2 \left( \frac{1}{2} \right) (7 \cos \left( \frac{5\pi}{24} \right))(7 \sin \left( \frac{5\pi}{24} \right)) = 23.66518 \text{ square inches} \\
\text{b) What is the area of the colored region (both red and yellow)?} \\
\text{Ans: } \left( \text{Fraction of circle} \right) \times \text{(area of circle)} = \\
\frac{5\pi}{24} (\pi 7^2) = \frac{245\pi}{24} = 32.0704 \text{ square inches.} \\
\text{c) What is the area of the yellow region?} \\
\text{Ans: } 32.0704 - 23.66518 = 8.40522 \text{ square inches.}
\end{align*}

16 Additional Problems for Exam III

1. What is the equation of the circle with center \((-2, 7)\) which is tangent to the line with equation \( y = \frac{3}{4} x + \frac{9}{4} \)?

\text{Ans: } \text{This is the line } 3x - 4y + 9 = 0 \text{ and the perpendicular distance to the line from } (-2, 7) \text{ is} \\
d = \frac{\left| 3(-2) - 4(7) + 9 \right|}{\sqrt{4^2 + 3^2}} = \frac{25}{5} = 5 \text{ so the radius is 5.}
2. If \( a = 5, b = 6, c = 7 \) (all in feet) then what is the measure of \( \angle A \)?

![Diagram of a triangle with labels A, B, C and sides a, b, c]

**Ans:** \( A = \arccos\left(\frac{5}{7}\right) = .775193733 \text{ rad} \), \( B = \arccos\left(\frac{19}{33}\right) = .99696087 \text{ rad} \)

3. The isosceles triangle is symmetric about the \( y \) axis, has an altitude of 10 inches, and the distance from \( B \) to \( C \) is 6 inches. The radius of the circle is 1 inch. What are the coordinates of the center of the circle and what are the coordinates of the points of tangency?

![Diagram of an isosceles triangle with a circle inscribed]

**Ans:** The slope of the line \( AC \) is \( \frac{10}{3} \) so the line has equation \(-10x + 3y = 0\). If the center of the circle is \((0, k)\) then the distance from \((0, k)\) to the line \( AC \) is the radius of the circle so \( \frac{|-10(0) + 3k|}{\sqrt{3^2 + (-10)^2}} = 1 \) so \( k = \frac{\sqrt{109}}{3} \).

The right point of tangency \((x, y)\) is on the line \( y = \frac{10}{3} \) and the circle so \((x-0)^2 + \left(\frac{10}{3}x - \frac{\sqrt{109}}{3}\right)^2 = 1\).

Use the quadratic formula to get \( x = .957 \).

4. The derivative in calculus computes the slopes of tangent lines. A standard calculus 1 problem asks for the angle made by two tangent lines. If the line \( A \) has slope \( \frac{7}{8} \) and the line \( B \) has slope \( \frac{2}{5} \) then what is the tangent of the acute angle made by the two lines?
**Ans** If $\alpha$ is the angle made by $A$ and the $x$ axis then $\tan(\alpha)$ is the slope of the line $A$. Similarly, if $\beta$ is the angle made by $B$ and the $x$ axis then $\tan(\beta)$ is the slope of $B$. If $\gamma$ is the angle through which $B$ must be rotated (counterclockwise) to come into coincidence with $A$ then looking at the triangle made by $A$, $B$, and the $x$ axis we have $\beta + \gamma + (\pi - \alpha) = \pi$ so $\gamma = \alpha - \beta$ and
\[
\tan(\gamma) = \tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} = \frac{\frac{7}{8} - \frac{2}{5}}{1 + \frac{7}{8}\cdot\frac{2}{5}} = \frac{19}{54}
\]
Exam III will be on the material on circles and introduction to trig covered in the lectures and homework.

Topics are:

1. Define the locator point of a number $x$ as a point on the unit circle.
2. Define the radian measure of an angle $\angle AOB$ where $O$ is the origin and $A$ is on the positive $x-$axis.
3. Convert between radian and degree measure.
4. Define $\sin(x)$ and $\cos(x)$ using the locator point of $x$.
5. Know and be able to use the geometric definition of the circle of radius $r$ with center $P$ as the set of all points whose distance from $P$ is $r$.
6. Calculate the length of an arc given the radian measure and the radius.
7. Calculate the radius given the radian measure and the length of the subtended arc.
8. Calculate the area of a sector given the arc and radius or the angle measure and radius.
9. Use a table of sines and cosines and the trig identities to calculate the sines, cosines, and tangents of the sum and difference and of half of the angles in the table.
10. Understand what has happened if the calculation of the tangent of an angle leads to an undefined number.
11. Know how to use the quadrant that an angle is in to choose the correct sign when calculating $\sin$ or $\cos$ using the half the angle formulas.
12. Calculate the angle between the hands of an analog clock, given the time.
13. Calculate the equation of a circle with a given line segment $\overline{AB}$ as a diameter using the formula $(x - a_1)(x - b_1) + (y - a_2)(y - b_2) = 0$ where $A = (a_1, a_2)$ and $B = (b_1, b_2)$.
14. Calculate the cartesian equation for a circle of radius $r$ with center $(h, k)$.
15. Determine the radius and center of a circle when given the cartesian equation in expanded form (i.e. $x^2 + y^2 + ax + by + c = 0$).
16. Calculate the equation of the line through the points of intersection of a circle.
17. Calculate the points of intersection of two circles.
18. Calculate the points on one line that are a given distance from another line using the formula for distance from a point to a line.

19. Calculate the points of intersection of a circle and a parametric line.

20. Complete a table of values of sines, cosines and tangents of the multiples of $\pi/6$ and $\pi/4$.

21. Solve triangles and problems involving triangles with law of cosines and law of sines.

22. Know Euler’s formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

23. Know and be able to use the following identities:

   (a) $\sin^2(x) + \cos^2(x) = 1$
   (b) $\sin(-a) = -\sin(a)$
   (c) $\cos(-a) = \cos(a)$
   (d) $\sin(\pi - a) = \sin(a)$
   (e) $\cos(\pi - a) = -\cos(a)$
   (f) $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
   (g) $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

   Know how to use the fact (f), (g), and the one for $\tan(a + b)$ given below are identities to get the corresponding formulas for $\sin(a - b)$, $\cos(a - b)$, and $\tan(a - b)$.

   (h) $\sin(2a) = 2\sin(a)\cos(a)$
   (i) $\cos(2a) = \cos^2(a) - \sin^2(a)$
   (j) $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$
   (k) $a^2 = b^2 + c^2 - 2bc\cos(A)$

Know and be able to use these formulas which will be written on the test.

   (a) $\tan(a + b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$
   (b) $\cos\left(\frac{a}{2}\right) = \pm \sqrt{\frac{1 + \cos(a)}{2}}$, and $\sin\left(\frac{a}{2}\right) = \pm \sqrt{\frac{1 - \cos(a)}{2}}$
   (c) $d = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}}$
17 November 20

1. If \( \triangle ABC \) is a triangle such that \( AB = 4, BC = 3, \) and \( AC = 5 \) then what are the angles \( \angle CAB, \angle CBA, \) and \( \angle ABC? \)

![Triangle ABC diagram]

**Ans:** \( a^2 = b^2 + c^2 - 2bc \cos(A) \)
\[
\cos(A) = \frac{3}{5}
\]
\[
\text{arccos}(\frac{3}{5}) = .92729 \text{ rad.}
\]
\[
A = .92729 + 2\pi n \text{ or } A = -.92729 + 2\pi n
\]
\[
0 \leq A \leq \pi \text{ so only } A = .92729 \text{ is possible.}
\]

2. If \(-3 \leq x \leq 12\) and \( \cos(x) = .7 \) then what are the possible values of \( x? \)

**Ans:** \( \text{arccos}(.7) = .79539 \) (rad.) so \( x = .7 + 2\pi n \) or \( x = -.7 + 2\pi n \)

- \(-3 \leq .7 + 2\pi n \leq 12 \) for \( n = -1, 0, 1 \)
- \(-3 \leq \pi - .7 + 2\pi n \leq 12 \) for \( n = 0, 1, 2 \)

3. The slope of the line in the diagram is 3.4. What is the acute angle \( \alpha \) that the line makes with the \( x-axis? \)

![Line diagram]

**Ans** \( \text{arctan}(3.4) = 1.2847 \). so \( \alpha = 1.2827 + 2\pi n \) or \( \alpha = 1.2827 + \pi + 2\pi n \)

Acute angles are between 0 and \( \frac{\pi}{2} \) so \( 1.2827 = 73.671^\circ \) is the only choice.

4. If angle \( B = \frac{\pi}{8}, \) \( a = 13 \) and \( b = 6 \) then what are the possibilities for the measure of \( A? \)
\[ \sin(A) = \frac{\sin\left(\frac{\pi}{8}\right)}{6} \]

\[ \sin(A) = 13 \frac{38268}{6} = .8291 \]

\[ A = \arcsin(.8291) + 2n\pi \text{ or } A = \pi - .8291 + 2n\pi \]

\[ 0 \leq A \leq \pi - \frac{\pi}{8} = 2.7488 \]

\[ A = .8291 \left(47^o\right) \text{ or } A = 2.31249 \left(132.4960692^o\right) \]

5. Let \( f(x) = x^2 - 2x + 6 \) and \( f_1(x) \) the restriction of \( f \) to the interval \((-\infty, 1]\) and \( f_2 \) the restriction of \( f \) to the interval \([1, \infty)\).

(a) What are the domains and ranges of \( f_1 \) and \( f_2 \)?
(b) Explain why \( f_1 \) and \( f_2 \) are one-to-one (injective).

\[ \text{Ans: If } a \text{ and } b \text{ are in the domain of } f_1 \text{ then } a \leq 1 \text{ and } b \leq 1. \text{ If } f_1(a) = f_1(b) \text{ then } a^2 - 2a + 6 = b^2 - 2b + 6 \text{ so } (a-b)(a+b) = 0 \text{ or } (a-b)(a+b-2) = 0. \text{ Since } a, b \] are both \( \leq 1 \) this can happen only if \( a = b \) or \( a = b = 1 \). A similar argument works for \( f_2 \).
(c) Find an explicit formula for \( f_1^{-1} \) and \( f_2^{-1} \)

\[ \text{Ans: Interchange the roles of } x \text{ and } y \text{ and use the quadratic formula. There will be two possibilities: one for } f_1 \text{ and one for } f_2 \]

(d) Write \( f_2^{-1} \) in terms of \( f_1 \)

\[ \text{Ans: } f_1^{-1} + f_2^{-1} = c, \text{ a constant. Determine } c \text{ and write } f_1^{-1} = c - f_1^{-1} \]

18 December 2

1. Let \( P(t) \) be the position at time \( t \) seconds of a particle in counterclockwise, uniform circular motion around the circle of radius \( r \) inches, with center \( C(h, k) \) (also in inches) and initial angular position 2 radians. If the particle makes one full circuit every 5 seconds then \( P(t) = \) ________________.

\[ \text{Ans: } P(t) = (h + r \cos(2 + \frac{2\pi}{5}t), k + r \sin(2 + \frac{2\pi}{5}t)) \]

2. The large hand of a clock measures 6 inches and small hand 4 inches.

(a) The radial velocity of the large hand is ________________.
Ans: $-2\pi \frac{\text{rad}}{\text{hr}}$

(b) The radial velocity of the small hand is ________________.

Ans: $\frac{\pi \text{ rad}}{6 \text{ hr}}$

(c) The end of the large hand is moving around a 6-inch circle at a rate of ________________.

Ans: $2\pi \frac{\text{rad}}{\text{hr}} \times \frac{6 \text{ in}}{\text{radius}}$ clockwise

(d) The radial velocity of the small hand is ________________.

Ans: $\frac{\pi \text{ rad}}{6 \text{ hr}} \times \frac{6 \text{ in}}{\text{radius}}$

3. At time $t$ hours the position in miles of a car moving at constant velocity around a circular race track is $P(t) = (1 + 3 \cos(\frac{\pi}{2} - 4\pi t), 2 + 3 \sin(\frac{\pi}{2} - 4\pi t))$

(a) The car moves in the __________ direction.

Ans: clockwise

(b) The car makes a circuit of the track every __________ minutes.

Ans: 30

(c) In 10 hours the car will have traveled __________ miles.

Ans: $4\pi \frac{\text{rad}}{\text{hr}} \times 10 \text{ hr} \times \frac{3 \text{mi}}{\text{rad}}$

19 December 4

1. The equation of the tangent line to the ellipse with equation $5x^2 + 4y^2 - 56 = 0$ at the point $(2, 3)$ is $y = ________$.

Ans: $20x + 24y - 112 = 0$

2. If $f(x, y) = 13 + 3x - 3x^2 - y$ then the equation of the tangent line to the graph of $f(x, y) = 0$ at the point $(2, 7)$ is ________.

Ans: $-9x + 25 - y = 0$

3. The equation of the tangent line to the graph of $x^3 + 2x^2y - y^3 + 3$ at the point $(1, 2)$ is ________.
Ans: $11x - 10y + 9 = 0$

4. Calculate the tangent line to the algebraic curve $x^2 + 5xy + y^2 + 5 = 0$ at the point $P(-2, 1)$
   Ans: $x - 8y + 10 = 0$

5. Calculate the tangent line to the algebraic curve $x^5 - 15xy^2 + 3y^4 = 5$ at the point $P(2, 3)$
   Ans: $-55x - 322 + 144y = 0$

6. Suppose $f(x) = 2x + 3 + (x - 4)^2H(x)$ and $g(x) = -5x + 12 + (x - 4)^9Q(x)$. Calculate the equation of the tangent line to the graph of $y = f(x)g(x)$ at $x = 4$.

   Ans: $y = f(x)g(x)$
   $f(x)g(x) = (2x + 3 + (x - 4)^2H(x))(-5x + 12 + (x - 4)^9Q(x))$
   $y = (2x + 3)(-5x + 12) + (x - 4)^2L(x)$.

   Write $(2x + 3)(-5x + 12) + (x - 4)^2Q(x)$ in form $\alpha + \beta(x - 4) + \gamma(x - 4)^2$

   then the tangent line is $y = \alpha + \beta(x - 4)$ which is $y = -88 - 71(x - 4)$
1. Use Newton’s method, starting at \( x = -2 \) to approximately find a point on the graph of \( y = x^4 - 7x^2 - 5 \) where the slope of the tangent line is equal to 1.

**Ans:**

\[
\begin{align*}
f(x) &= x^4 - 7x^2 - 5 \\
g(x) &= 4x^3 - 14x - 1 \\
g'(x) &= 12x^2 - 14
\end{align*}
\]

\[
\begin{align*}
x_{n+1} &= x_n - \frac{g(x_n)}{g'(x_n)} \\
x_{n+1} &= x_n - \frac{4x_n^3 - 14x_n - 1}{12x_n^2 - 14}
\end{align*}
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( g(x_n) )</th>
</tr>
</thead>
<tbody>
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2. Go to the Supplemental Materials where you will find the keys to the exams, plus the study guide for Exam 3.

The quiz Wed. will contain a problem from Exam 1.