Monday, April 25, 2016:

- Final Exam next week → Different room & time!
- Tell me about exam conflicts via email now!
- Final Exam is cumulative!

Q: How many people can live on the Earth?

(Flawed) estimate of upper bd. Surface area of Earth ≈ 510,072,100 km² (land + H₂O)

= 510,072 × 10⁹ m². If each person gets 1 sq meter of room.
time

unbounded, i.e., population

\[ y = C e^{kt} \]

\[ \frac{dy}{dt} = ky(t) \]

Population growth constant: \( y(t) = pop \)

3.4 \cdot 10^5
A better model:

Let \( A > 0 \) be the "carrying capacity" for a system, and \( y(t) \) the pop. at time \( t \). i.e. \( A \) is max possible population. The logistic equation assigns a growth constant \( k > 0 \) to the system, and models change in population by

\[
\frac{dy}{dt} = k \cdot y \cdot (1 - \frac{y}{A}).
\]

This factor inhibits pop. growth. As \( y \to A \), \( 1 - \frac{y}{A} \to 0 \) leads to usual exponential growth.
\[ y(t) \overset{I}{\leftrightarrow} \mathcal{L}^{-1} \left\{ \frac{At}{s + A} \right\} = \frac{A}{s + A}. \]

So,
\[ e^{-kt} \overset{\text{as } t \to \infty}{\to} 0. \]

Then, occurrence of \( t \) is \( y(t) \).

As \( t \to \infty \), what happens to \( y(t) \)?

\[ y(t) = y \]

\[ y(0) = \text{initial}\]

\[ \frac{(0)}{A} - \left( \frac{y(0)}{A} \right) + \frac{t}{A} (y(0) - A) \]

\[ 1 - e^{-kt} \]

\[ \frac{y(t)}{A} = \frac{\text{var}}{A} \]

\[ \text{therefore} \]

\[ \frac{y(t)}{A} = \frac{2t}{A} \text{ for } t \geq 0 \text{ and } 2t < A. \]
Note:

- If \( y = 0 \), we get an **unstable equilibrium**, i.e., \( f \) in \( y(t) \) is constant, but if \( y(0) \) is close to 0, it moves away from 0 as time increases, either to \(-\infty\) or \( \infty \).

- If \( y = A \), we get a **stable equilibrium**, because \( y(t) = A \) is an equilibrium solution if \( y(0) \) is close to \( A \), it gets closer as time increases.

- If \( y(0) > 0 \), then \( y(t) \to A \) as \( t \to \infty \).
Proof of Thm: Separate Variables.

\[ \frac{dy}{y(1-x)} = k \, dx \, t \]

Goal is to integrate and solve.

RHS: \[ \int k \, dx = ke + C \sim \text{const.} \]

LHS: \[ \int \frac{1}{y(1-x)(y-A)} \, dy = \int \left( \frac{1}{y} - \frac{1}{y-A} \right) \, dy \]

= \ln|y| - \ln|y-A| + \text{const.} \]
So, using log laws & combining constants:

\[ \ln \left| \frac{Y}{Y-A} \right| = kt + C \]  
\[ \text{New constant.} \]

\[ \Rightarrow \left| \frac{Y}{Y-A} \right| = e^{kt+C} = e^c \cdot e^{kt} \]

\[ \Rightarrow \frac{Y}{Y-A} = (\pm e^c) \cdot e^{kt} \]
\[ \text{New constant, call it } C \text{ again.} \]

\[ \Rightarrow \frac{Y}{Y-A} = C \cdot e^{kt} \]
Set $t = 0$. We get

\[
\frac{y(0)}{y(0) - A} = C \cdot e^{k \cdot 0} = C.
\]

So,

\[
y(t) = \frac{A}{1 - \frac{T}{y(0) - A}} \cdot e^{kt}
\]

\[
= \frac{1}{A} - \frac{T}{y(0) - A} \cdot e^{kt}
\]
Thought exercise:

Q: How might you solve
\[ \frac{dy}{dt} = k \cdot y (y-2)(y-3) \, ? \]

Q: Can you actually solve for \( y \) here?
(I don't know...)
Ex: Suppose $A = 2000$, $k = 0.6$.

Q1: What is $y(t)$ if $y(0) = 500$?

Q2: If $y(0) = 500$, how long until $y(t) = 1000$?

A1: $y(t) = \frac{A}{1 - e^{-kt}(\frac{y(0) - A}{y(0)})} = \frac{2000}{1 + e^{-0.6t}}$

A2: $1000 = \frac{2000}{1 + 3e^{-0.6t}}$, solve for $t$,

get $t = \frac{-10}{6} \ln \left( \frac{1}{3} \right) \approx 1.85$ years assuming units.