How can we add infinitely many terms together?

First big question (from 9th grade):

\[ \frac{1}{0.1} + \frac{1}{0.01} + \frac{1}{0.001} + \cdots = \frac{1000000}{100\times100\times100} + \frac{100000}{100\times100} + \frac{1000}{100} + \frac{10}{1} = \frac{10}{1} = 10 \]

or

\[ \frac{7}{4} = 0.875 \]

or

\[ \frac{11}{8} = 1.375 \]

or

\[ \frac{11}{8} = 1.375 = 1.375 \]

\[ = 0.75 + 0.75 + 0.75 + \cdots = 0.75 \]

G: What about 0? What about 1?

Work & discuss with your neighbors! Thing near.

\[ \frac{1}{b} + \frac{1}{b} + \frac{1}{b} + \frac{1}{b} = \frac{4}{b} = 0.999999999999 \]

\[ \frac{1}{b} = 0.9 = 0.999999999999 \]

\[ 0 \]

Do you believe that?
• Start w/ sequence of numbers:
  \[ a_1, a_2, a_3, a_4, \ldots = \sum_{n=1}^{\infty} a_n \]
  \[ \frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \ldots = \sum_{n=1}^{\infty} \frac{9}{10^n} \]

• Create a partial sum:
  \[ a_1, a_1+q_2, a_1+q_2+q_3, \ldots \]
  examples.

• Create an infinite series:
  \[ a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n \]
  \[ \text{define } S_N = a_1 + \ldots + a_N \]

\[ \Sigma \] means \[ \text{sum} \]
\[
\sum_{k=1}^{\infty} \frac{9}{10^k} = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots = 0.9999\ldots
\]

\( T = 0 \)

\( 1 + \frac{9}{16} + \frac{9}{16^2} + \ldots = 1.3\overline{5} 

\text{Ex: } \frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \frac{16}{3} + \ldots = 1.3\overline{5} 

\text{Sect: } \frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \ldots

\text{Partial Sums: } \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = S_n

\text{Infinite Series: } 1 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = 0.9999\ldots

10\%
Read 10.1 for recitation next week!

Focus on bounded and monotonic sequences — we will discuss in lecture recitation (except Math Excel) Wed., after you grapple with them in class. Course Policy Applies to me + TA's as well as students. All won't.

We're all needing to improve.

Exercise: Write \( \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{16}, \ldots \)

\( S = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{16} + \ldots \)

1. \( \frac{1}{3} \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} \)
2. a recursion \( a_{n+1} + a_n = a_n + a_{n+1} \)
3. a recursion \( a_n = \frac{1}{n+1} \)

\( 1, 1, 2, 1, \ldots, 5, 5, 8, 13, \ldots \)
Two Messy Examples:

A) \( \{ a_n \}_{n=1}^{\infty} = \{ 1, -1, 1, -1, 1, -1, ... \} \)

\[ 1 - 1 + 1 - 1 + 1 - 1 + \cdots = ? \]

\[ S_1 = a_1 = 1 \]
\[ S_2 = a_1 + a_2 = 0 \]
\[ S_3 = a_1 + a_2 + a_3 = 1 \]
\[ S_4 = a_1 + a_2 + a_3 + a_4 = 0 \]
\[ S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1 \]
\[ \vdots \]
\[ S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n \]

\[ S_N = 1, 0, 1, 0, 1, 0, \ldots \]

bad limiting behavior

B) \( \{ a_n \}_{n=1}^{\infty} = \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \} = \{ \frac{1}{n} \}_{n=1}^{\infty} \)

Infinite series: \[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \]

"Harmonic Series"

Super complicated and subtle.
Q: What is a good defn of "limiting behavior"?

Defn: Say that \( \sum_{n=1}^{\infty} q_n \) converges to a limit \( L \), written \( \lim_{n \to \infty} q_n = L \) or \( q_n \to L \) as \( n \to \infty \), if for every \( \varepsilon > 0 \) there is a number \( M \) so that \( |q_n - L| < \varepsilon \) for all \( n > M \).

If \( \sum_{n=1}^{\infty} q_n \) has no limit, it diverges. If it grows w/out bound, it diverges to infinity.

\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & \ldots & M
\end{array} \]
Def: An infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$ converges to a sum $S$ if the partial sums converge to $S$, i.e.

$$\lim_{N \to \infty} S_N = S.$$ 

We write $S = \sum_{n=1}^{\infty} a_n$. If limit does not exist, $\sum_{n=1}^{\infty} a_n$ diverges. If limit is infinite, say $\sum_{n=1}^{\infty} a_n$ diverges to $\infty$. 


What is \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \)?

\[
S_1 = \frac{1}{2} \\
S_2 = \frac{3}{4} \\
S_3 = \frac{7}{8} \\
S_4 = \frac{15}{16} \\
S_5 = \frac{31}{32}
\]

\[S_N = \frac{2^N - 1}{2^N}\]

\[\lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{2^N - 1}{2^N} = 1 - 0 = 1.\]

To "formally" show this, use induction.
why is \(0.9999\ldots = 1\) ?

\[\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = \]

\[9 \left( \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \ldots \right) = \]

\[9 \left( \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots \right) = \]

\[9 \left( \left( \frac{1}{10} \right)^1 + \left( \frac{1}{10} \right)^2 + \left( \frac{1}{10} \right)^3 + \left( \frac{1}{10} \right)^4 + \ldots \right) = \]

what if we used \(\frac{1}{3}\) ? \(\frac{7}{8}\) ?

\(\infty 1?\) \(1+1+1+1+1+\ldots\)