Wed, Jan 20, 2016 10. Quiz in recitation tomorrow!

1. Lecture notes are posted on my 114 site.

2. Attendance sign-in sheets start Friday; they will be on desk by doors.

3. Remember: Our first big Q is “How do we add up infinitely many items?”

Before class: Talk with your neighbors:

do you believe it is possible to add up infinitely many values and get a finite answer? Why or why not?

Examples: \[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1 \]

or \[ \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots = \overline{0.9} = 1 \]
Thm 4, §10.1: If $f(x)$ is continuous, and
\[ \lim_{n \to \infty} a_n = L, \text{ then } \]
\[ \lim_{n \to \infty} f(a_n) = f\left( \lim_{n \to \infty} a_n \right) = f(L). \]

Example: \[ \lim_{n \to \infty} \left(2 + \frac{4}{n^2}\right)^{\frac{1}{3}} \]

What is $a_n$?
What is $f$?
What is $\lim_{n \to \infty} a_n$?

\[ a_n = \frac{4}{n^2} \]
\[ q_n = 2 + \frac{4}{n^2} \]
Both of these are reasonable choices.

Use this...

\[ \lim_{n \to \infty} \frac{4}{n^2} = 0 \]
\[ \lim_{n \to \infty} 2 + \frac{4}{n^2} = 2 \]
\[ \Rightarrow f(x) = x^{\frac{1}{3}} \]
So,
\[ \lim_{n \to \infty} \left(2 + \frac{4}{n^2}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} = 2 \]
Def: A sequence $\{a_n\}$ is:

(bounded from above (below) if there is a number $M$ such that $a_n \leq M$ ($M \geq a_n$) for all $n$.

If $\{a_n\}$ is bounded from both above and below, say it is bounded. Otherwise, it is **unbounded**.

Thm 5 in §10.1: Convergent sequences are bounded.

**Idea:**

\[ a_0, a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots \]

 bounds

\[ 1, 2, 3, 4, 5, 6, 7, \ldots \]

**Note:** Converse not true! $\{1, -1, 1, -1, 1, -1, \ldots\}$ is bounded, but does not converge.
...not be your plan might...

...monotonic sequence that does not converge.

Thm 6.19.1: If $\{a_n\}$ is a bounded and bounded from below by zero.

And this is monotonically decreasing.

Always either (i) increasing or (ii) decreasing.

Def: A sequence $\{a_n\}$ is monotonic if $a_n$ is...

Ex: $a_n = \frac{n-5}{16}$ for $n = 6, 7, 8, ...$
This example is super, super important!

→ This is used to prove almost everything in this course.

A geometric sequence is: for real number r, and real number c > 0.

The sequence is \( c, cr, cr^2, cr^3, cr^4, \ldots \).

**Ex:** \( c = 9, r = \frac{1}{10}, 9, \frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \frac{9}{10000}, \ldots \).

\( c = 72, r = \frac{1}{109}, 72, \frac{72}{109}, \frac{72}{1001}, \frac{72}{100001}, \ldots \).

A finite geometric series is a partial sum of the form \( c + cr + cr^2 + cr^3 + \ldots + cr^n \), for some \( N \).

A geometric series is \( c + cr + cr^2 + cr^3 + \ldots \).
Thm: \[ c + cr + cr^2 + cr^3 + \ldots + cr^n = c \left( \frac{1 - r^{n+1}}{1 - r} \right) \]

- If \( c \neq 0 \), and \( |r| < 1 \), then
  \[ \sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r} \]

**Pf:** Look in text, §10.2, or look at similar triangle "proof w/out words" link on my website.

**Ex:** \( c = 9, r = \frac{1}{10} \).

\[ 9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = \frac{9}{1 - \frac{1}{10}} = \frac{9}{\frac{9}{10}} = 10. \]

Subtract 9 from both sides, we get \( .9 = 1 \).

**Exercise:** \( c = 72, r = \frac{1}{100} \) \( \Rightarrow \frac{72}{100} = .72 \).
Mon, Jan 25, 2016

0. SIGN IN AT SHEETS ON FRONT TABLE!!

1. See email from UK Canvas system for schedule change.

2. No office hours Fri. Jan 29. Make-up hour on Mon, Feb 1, from 12-12:15.

3. Section 019: TA Change: Your new TA is Yaowei Zhang, starting tomorrow.

4. With your neighbors, compute partial sums for $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Use calculator or phone.
Geom. Series
\[ c + cr + cr^2 + cr^3 + \cdots + cr^N = c \frac{1 - r^{N+1}}{1 - r} \]

\[ [c > 0, \ -1 < r < 1] \]

Pf by convincing example: \( N = 5 \)
\[
(c + cr + cr^2 + cr^3 + cr^4 + cr^5)(1-r) = \]
\[ c + cr^2 + cr^3 + cr^4 + cr^5 \]
\[ \underline{-cr - cr^2 - cr^3 - cr^4 - cr^5} = \]
\[ c - cr^6 = c(1 - r^6) \]

Divide both sides by \( 1-r \), I get the formula.
\[
\begin{align*}
\lim_{n \to \infty} \left( \frac{c + cr + cr^2 + \cdots}{c(1 - r)} \right) &= 0, \quad \text{since } 0 < r < 1 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c(1 - r)} &= 0 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 0 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c(1 - r)} &= 0 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= \infty \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 1 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 0 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 1 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 0 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 1 \\
\Rightarrow \quad \lim_{n \to \infty} \frac{c + cr + cr^2 + \cdots}{c} &= 0
\end{align*}
\]
\[ cr + cr^2 + cr^3 + \ldots = \frac{cr}{1-r} \]

\[ c + cr + cr^2 + \ldots = \frac{c}{1-r} \]

Multiply by \( r \)
\[ \pi = \frac{C}{D} \]

\[ \pi \approx 3.14 \]

A little bit \( \frac{1}{7} \)

Nasty + complicated
Telescoping Series: "Leibniz Series"

\[
\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots = 1.
\]

\[
\sum_{n=1}^{N} \frac{1}{n(n+1)} \quad \text{Fact:} \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.
\]

\[
S_N = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{N(N+1)} =
\]

\[
= \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{N-1} - \frac{1}{N} \right) + \left( \frac{1}{N} - \frac{1}{N+1} \right)
\]

\[
= \frac{1}{1} - \frac{1}{N+1} \quad \lim_{N \to \infty} \left( 1 - \frac{1}{N+1} \right) = 1 - 0 = 1.
\]
Thm 1 in §10.2: If \( \sum_{n=1}^{8} a_n, \sum_{n=1}^{8} b_n \) converge, then so do \( \sum_{n=1}^{8} (a_n + b_n) \).

These also converge.

Q: What is \( a_n, b_n \) for this series?

\[
\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \frac{2+3^0}{5^0} + \frac{2+3^1}{5^1} + \frac{2+3^2}{5^2} + \ldots
\]

\[a_n = \frac{2^n}{5^n} = 2(\frac{1}{5})^n, \quad b_n = \frac{3^n}{5^n} = (\frac{3}{5})^n\]
\[
\text{NOTE: } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2 \left( \frac{1}{5} \right)^n = 2 \sum_{n=1}^{\infty} \left( \frac{1}{5} \right)^n \\
a_1 + a_2 + a_3 + \cdots = \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots = \\
2 \left( \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots \right) = 2 \cdot \frac{\frac{1}{5}}{1 - \frac{1}{5}}
\]

\[
\frac{1}{5} \left( 1 + \frac{1}{5} + \frac{1}{5^2} + \cdots \right) = \frac{1}{1 - \frac{1}{5}}
\]

\text{NOTE: Similarly add up bn geom. series, then add the total sum.}
Ex: \( 1-1+1-1+1-1+\ldots \) diverges.
Partial sum sequence is \( 1, 0, 1, 0, 1, 0, \ldots \), \( \overline{\text{no limit}} \)

Divergence Test: if \( a_n \to 0 \), then 
\[ a_1 + a_2 + a_3 + \ldots \] diverges.

→ See book for a proof.

Ex: \( \sum_{n=1}^{\infty} \frac{7n}{8n+2} = \frac{7}{8} + \frac{7.2}{8.2+2} + \frac{7.3}{8.3+2} + \ldots \)

\[ \lim_{n \to \infty} \frac{7n}{8n+2} = \lim_{n \to \infty} \frac{7}{8 + \frac{2}{n}} = \frac{7}{8}. \quad \text{So,} \]
\[ \text{divides by n num + den} \]
Harmonic Series:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n} \]

Thm: The harmonic series diverges.