Monday, March 21, 2016

→ SIAM student talk Tues 5-6 PM

§7.6: Improper Integrals

Ex: \[ \int_1^R \frac{1}{x^4} \, dx = \int_1^R x^{-4} \, dx = \left[ \frac{1}{-3} x^{-3} \right]_1^R = \frac{1}{3} - \frac{1}{3R^3} . \]

So this is positive.

R > 1 implies \[ \frac{1}{3R^3} < \frac{1}{3} . \]

We define \[ \int_1^\infty \frac{1}{x^4} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^4} \, dx = \frac{1}{3} . \]

Since \[ \frac{1}{3R^3} \to 0 \] as \( R \to \infty \).
**Def.** Fix a in real $\mathbb{R}$ and assume $f$ is integrable on $[a,b]$ for all $b > a$.

We define $\int_a^b f(x) \, dx = \lim_{R \to \infty} \int_a^R f(x) \, dx$.

Similarly, write $\int_{-\infty}^a f(x) \, dx = \lim_{R \to -\infty} \int_R^a f(x) \, dx$ if $f$ is integrable on $[R,a]$ for all $R$.

If $f$ is cts on $(a,b]$ but discontinuous at $a$,

define $\int_a^b f(x) \, dx = \lim_{R \to a^+} \int_R^b f(x) \, dx$. Similar for when $f$ is discts at $b$. 


Q: What about $\frac{1}{xp}$ on $(0, \infty)$?

Reminder: \[
\int_1^x \frac{1}{t^p} dt = \ln(x).
\]

Thm: For $a > 0$, \[
\int_a^\infty \frac{1}{xp} \, dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}
\]

\[
\int_0^a \frac{1}{xp} \, dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}
\]

Important and complicated.

Opposite conditions.
Ex: Compute: \( \int_{2}^{\infty} \frac{1}{x^7} \, dx \) using (i) Thm (ii) limit defn.

Since the integral goes to \( \infty \) and \( p = 7 > 1 \), we get

\[
\int_{2}^{\infty} \frac{1}{x^7} \, dx = \frac{2^{-6}}{6} = \frac{2^{1-7}}{7-1}.
\]

As a limit, \( \lim_{R \to \infty} \int_{2}^{R} \frac{1}{x^7} \, dx = \lim_{R \to \infty} \left[ \frac{x^{-6}}{-6} \right]_{2}^{R} = \lim_{R \to \infty} \left[ \frac{R^{-6}}{-6} - \frac{2^{-6}}{-6} \right] = \frac{2^{-6}}{-6}.
\]
Exercise: Complete other cases.

Thm: (Comparison Test for integrals):
Assume $f(x) \geq g(x) \geq 0$ for $x \geq a$, then

(i) If $\int_{a}^{\infty} f(x)\,dx$ converges, then $\int_{a}^{\infty} g(x)\,dx$ converges.

(ii) If $\int_{a}^{\infty} g(x)\,dx$ diverges, then $\int_{a}^{\infty} f(x)\,dx$ diverges.
Ex: Does \( \int_1^\infty \frac{1}{2\sqrt{x} + e^{2x}} \, dx \) converge?

Step 1: Find a comparison function for \( \frac{1}{2\sqrt{x} + e^{2x}} \).

Use \( \frac{1}{e^{ax}} \).

\[
\frac{1}{\sqrt{x}} \cdot \frac{1}{e^{2x}} \quad \text{\( \leq \)} \quad \text{Suggestion...}
\]

\[
\frac{1}{2\sqrt{x} + e^{2x}} \leq \frac{1}{e^{2x}}
\]

for \( x \geq 1 \).

\[
\int_1^\infty \frac{1}{e^{2x}} \, dx = \lim_{R \to \infty} \int_1^R \frac{1}{e^{2x}} \, dx = \lim_{R \to \infty} \frac{1}{2}(e^{-2} - e^{-2R})
\]

\[= \frac{1}{2} e^{-2} \]

Since \( \int_1^\infty \frac{1}{e^{ax}} \, dx \) converges, the comparison test implies the original integral converges.
Application to infinite series:

§10.3: Thm (The Integral Test):

Let \( a_n = f(n) \) where \( f(x) \) is positive, decreasing, and \( cts \) for \( x \geq 1 \).

(a) If \( \int_1^\infty f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

(b) If \( \int_1^\infty f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

Ex: \( f(x) = \frac{1}{x} \), then \( a_n = f(n) = \frac{1}{n} \). So,

\[
\int_1^\infty \frac{1}{x} \, dx = \lim_{R \to \infty} \ln(R) = \infty. \text{ Thus, } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}
\]
Q: When we define $\Pi = \frac{C}{D}$, how do we actually measure $C$ accurately? i.e., How do we know $\frac{C}{D}$ makes sense as a ratio?
General problem: Given a differentiable curve $y = f(x)$, measure its length.

Example: $y = \sqrt{1-x^2} = f(x)$

Schematic Picture:

Goal: measure length of orange segment.

Mean Value Theorem: There is a $c_i$ in $(x_{i-1}, x_i)$ so that $f(x_i) - f(x_{i-1}) = \frac{f'(c_i)(x_i - x_{i-1})}{x_i - x_{i-1}}$.
From this, we have setting \( x_i - x_{i-1} = \Delta x_i \),
\[
f(x_i) - f(x_{i-1}) = f'(c_i) \Delta x_i.
\]

So, length of segment is \( \sqrt{(\Delta x_i)^2 + (f'(c_i)\Delta x_i)^2} \)

= \( \sqrt{1 + [f'(c_i)]^2} \cdot \Delta x_i \).

Approximate length of curve is
\[
\sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} \Delta x_i.
\]
Taking limit as \( n \to \infty \), we get

Length of curve is

\[
\int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

Since \( \Delta x_i \to ^{"dx"} \).

**Defn.** If \( f'(x) \) exists and is cts on \([a,b]\),
the arc length of \( f(x) \) over \([a,b]\) is

\[
\int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]
Ex: \( f(x) = mx + c \), i.e. straight line.

Using distance formula, length is

\[
\sqrt{(b-a)^2 + (m(b-a))^2} = (b-a)\sqrt{1+m^2}.
\]

Length is also

\[
\int_a^b \sqrt{1+f'(x)^2} \, dx = \int_a^b \sqrt{1+m^2} \, dx
\]

\[
= \sqrt{1+m^2} \int_a^b \, dx = (b-a)\sqrt{1+m^2}.
\]
Ex: What about a unit semicircle? Arc length should be $\pi$.

$$f(x) = \sqrt{1 - x^2}, \quad f'(x) = -\frac{x}{\sqrt{1 - x^2}}$$

So, $S = \int_{-1}^{1} \sqrt{1 + \left(\frac{x}{\sqrt{1 - x^2}}\right)^2} \, dx = \int_{-1}^{1} \sqrt{1 + \left(\frac{x}{\sqrt{1 - x^2}}\right)^2} \, dx = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx$.

The length of the semicircle is $\pi$. Therefore, the arc length is $\pi$.
\[
\frac{d}{dx} \arcsin(x) \bigg|_{-1}^{1} = \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\]

Domain for \( \arcsin(x) \) is \([-1, 1]\).

Range is \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

Ex: Catenary Curve: \( f(x) = a \cosh\left(\frac{x}{a}\right) \).

where \( \cosh(x) = \frac{e^x + e^{-x}}{2} \).
Suppose a chain hangs w/equation $f(x) = 2 \cosh \left( \frac{x}{a} \right)$ between $x = -1$, $x = 1$. What is length?

Two options: #1: Find $\frac{d}{dx} \cosh(x)$ in book & memorize it. Also learn hyperbolic trig
pyth. identity.

#2: Use $\frac{e^x + e^{-x}}{2}$ be clever.
#2: \[ \frac{d}{dx} \left( 2 \cosh \left( \frac{x}{2} \right) \right) = \frac{d}{dx} \left( 2 \cdot \frac{e^{x/2} + e^{-x/2}}{2} \right) = \frac{1}{a} \left[ e^{x/2} - e^{-x/2} \right] \left( = 2 \sinh \left( \frac{x}{2} \right) \right). \]

arc length is \[ \int_{-1}^{1} \sqrt{1 + \left( \frac{e^{x/2} - e^{-x/2}}{2} \right)^2} \, dx = \text{clever!} \]

\[ \int_{-1}^{1} \frac{e^{x/2} + e^{-x/2}}{4} \, dx = e^{1/2} - e^{-1/2}. \]

\[ \text{integral} \left\{ \text{do this}. \right\} \]