1. **Theorem 2.9 (Fundamental Theorem of Arithmetic - Uniqueness Part)**

Let \( n \) be a natural number. Let \( \{p_1, p_2, \ldots, p_m\} \) and \( \{q_1, q_2, \ldots, q_s\} \) be sets of primes with \( p_i \neq p_j \) and \( q_i \neq q_j \) if \( i \neq j \). Let \( \{r_1, r_2, \ldots, r_m\} \) and \( \{t_1, t_2, \ldots, t_s\} \) be sets of natural numbers such that

\[
n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} = q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}.
\]

Then \( m = s \) and \( \{p_1, p_2, \ldots, p_m\} = \{q_1, q_2, \ldots, q_s\} \). That is, the sets of primes are equal but their elements are not necessarily listed in the same order; that is \( p_i \) may or may not equal \( q_i \). Moreover, if \( p_i = q_j \) then \( r_i = t_j \). In other words, if we express the same natural number as a product of powers of distinct primes, then the expressions are identical except for the ordering of the factors.

**Proof:** We will prove this by (strong) induction on \( n \geq 2 \).

First check the base case: If \( n = 2 \), then the only way to write \( n \) as a product of primes is 2 itself.

For the inductive step, assume that the theorem is true for integers \( n = 2, \ldots, k \). Now suppose \( n = k + 1 \) and

\[
n = p_1^{r_1} \cdots p_m^{r_m} = q_1^{t_1} \cdots q_s^{t_s}
\]

are two factorizations of \( n \) into primes. From the first expression we know \( p_1 | n \). Thus we know \( p_1 | q_1^{t_1} \cdots q_s^{t_s} \). By Lemma 2.8, \( p_1 = q_i \) for some \( i \). Let \( m = n/p_1 \), which is an integer.

If \( m = 1 \), then the two expressions for \( n \) can only consist of a single occurrence of \( p_1 \), so the two expressions are equal, which is what we want to show. So suppose \( m > 1 \), and we have

\[
m = p_1^{r_1-1} p_2^{r_2} \cdots p_m^{r_m} = q_1^{t_1} \cdots q_i^{t_i-1} \cdots q_s^{t_s}.
\]

(If either or both \( r_1 - 1 \) and \( t_i - 1 \) are now zero, drop these terms from the expressions in (2).) By the induction hypothesis, as \( m < n \), the same set of primes is involved in both factorizations in (2), and each prime in the factorization appears with the same power in both factorizations. Hence the same statement is true of the two factorizations in (1) (just multiply both sides by \( p_1 \)).

Therefore the statement is true for all \( n \geq 2 \) by induction.

2. **Theorem 2.12**

Let \( a \) and \( b \) be natural numbers greater than 1 and let \( p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} \) be the unique prime factorization of \( a \) and let \( q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s} \) be the unique prime factorization of \( b \). Then \( a | b \) if and only if for all \( i \leq m \) there exists a \( j \leq s \) such that \( p_i = q_j \) and \( r_i \leq t_j \).

**Proof:** First assume \( a | b \). Then there is an integer \( k \) such that \( ak = b \). By the uniqueness of prime factorization, the prime factorization of \( ak \) will equal the product of the prime factorization of \( a \) with that of \( k \) (though it is possible that \( k \) is simply 1), and this product will equal the prime factorization of \( b \). Thus, for each \( i = 1, \ldots, m \), if \( p_i^{u_i} \) appears in the prime factorization of \( a \) and \( p_i^{u_i} \) appears in the prime factorization of \( k \) (possibly with \( u_i = 0 \)), then \( q_j = p_i \) for some \( j \) and \( t_j = u_i \). Therefore \( r_i \leq t_j \).

Conversely, assume that for all \( i \leq m \) there exists a \( j \) such that \( p_i = q_j \) and \( r_i \leq t_j \). For each \( q_j \) let \( u_j \) equal the power of \( q_j \) in \( b \) minus the power of \( q_j \) (possibly zero) in \( a \). By the assumption each \( u_j \geq 0 \). Let \( k = q_1^{u_1} \cdots q_s^{u_s} \). Then by this construction, \( ak \) has the same prime factorization as \( b \), and thus \( ak = b \) and \( a | b \).