1. **Theorem 2.32**
   For all natural numbers \( n \), \((n, n + 1) = 1\).

   **Proof 1:** Observe that \( n \cdot (-1) + (n+1) \cdot 1 = 1 \). Then by Theorem 1.39, \( \gcd(n, n+1) = 1 \).

   **Proof 2:** Assume \( d \) is a natural number such that \( d|n \) and \( d|(n + 1) \). Then \( d|((n + 1) - n) \), so \( d|1 \). Thus \( d = 1 \) or \( d = -1 \). Therefore, the largest common divisor of \( n \) and \( n + 1 \) is \( 1 \).

2. **Theorem 2.33**
   Let \( k \) be a natural number. Then there exists a natural number \( n \) (which will be much larger than \( k \)) such that no natural number less than \( k \) and greater than \( 1 \) divides \( n \).

   **Proof:** Consider \( n = k! + 1 \). We claim that no natural number \( m \) between \( 2 \) and \( k \) divides \( m \). Otherwise, if such \( m \) divides \( n \) then \( m|(n - k!) \), as \( m|k! \). Thus \( m|1 \), which implies \( m = 1 \).

3. **Theorem 2.34**
   Let \( k \) be a natural number. Then there exists a prime larger than \( k \).

   **Proof:** If \( k \) is \( 1 \) then a larger prime is \( 2 \), and we are done. So assume \( k \geq 2 \). Consider the set of all primes \( p_1, p_2, \ldots, p_n \) that are less than or equal to \( k \). Let \( m = p_1p_2 \cdots p_n + 1 \). Then none of the \( p_1, p_2, \ldots, p_n \) divide \( m \); else such a \( p_i \) divides \( m - p_1p_2 \cdots p_n = 1 \), forcing \( p_i = 1 \). Therefore all primes in the prime factorization of \( m \) are greater than \( k \). We conclude that there must exist a prime greater than \( k \).

4. **Theorem 2.35 (Infinitude of Primes Theorem)**
   There are infinitely many prime numbers.

   **Proof:** Assume the number of primes is finite. Let \( S = \{p_1, p_2, \ldots, p_n\} \) be the set of all primes. Let \( m = p_1p_2 \cdots p_n + 1 \). By the argument in Theorem 2.34, \( m \) has a prime factor other than \( p_1, p_2, \ldots, p_n \). Thus we have found a prime not in the set \( S \), which is a contradiction. Therefore the number of primes is infinite.