1. **Theorem 3.27.**

   Let \( a, b, m, \) and \( n \) be integers with \( m > 0 \) and \( n > 0 \). Then the system
   
   \[
   \begin{align*}
   x & \equiv a \pmod{n} \\
   x & \equiv b \pmod{m}
   \end{align*}
   \]

   has a solution if and only if \( \gcd(n, m) | a - b \).

   **Proof:** Suppose the given system has a solution \( x \). Let \( d = \gcd(n, m) \). We know \( x - a \) is a multiple of \( n \), hence also a multiple of \( d \). Similarly \( x - b \) is a multiple of \( m \), hence also a multiple of \( d \). Thus \( a - b = (x - b) - (x - a) \) is also a multiple of \( d \). Therefore \( d | (a - b) \).

   Conversely, suppose \( \gcd(n, m) = d | (a - b) \). Thus \( d\ell = a - b \), for some integer \( \ell \). We want to construct a solution \( x \) for the given system of congruences. Any integer \( x = a + kn \), where \( k \) is an integer, will satisfy the first equation. So it remains to find an integer \( k \) so that \( x \) also satisfies the second equation. Thus we need to find \( k \) such that \( a + kn \equiv b \pmod{m} \), which is equivalent to
   
   \[
   kn \equiv b - a \pmod{m}.
   \]

   By Theorem 1.40 there exist integers \( y \) and \( z \) such that \( ny + mz = d \). In other words \( ny \equiv d \pmod{m} \). Multiply the previous equation through by \( -\ell \) to get \( n(-y\ell) \equiv (b - a) \pmod{m} \). Thus setting \( k = -y\ell \), we have that \( x = a - y\ell n \) satisfies the given system.

2. **Theorem 3.28.**

   Let \( a, b, m, \) and \( n \) be integers with \( m > 0 \) and \( n > 0 \) and \( \gcd(n, m) = 1 \). Then the system
   
   \[
   \begin{align*}
   x & \equiv a \pmod{n} \\
   x & \equiv b \pmod{m}
   \end{align*}
   \]

   has a unique solution modulo \( mn \).

   **Proof:** By Theorem 3.27 we know that the system has a solution, because \( \gcd(n, m) = 1 | a - b \). To show the solution is unique modulo \( mn \), assume that both \( x \) and \( x' \) satisfy the system
   
   \[
   \begin{align*}
   x & \equiv a \pmod{n} \\
   x & \equiv b \pmod{m}
   \end{align*} \quad \text{and} \quad \begin{align*}
   x' & \equiv a \pmod{n} \\
   x' & \equiv b \pmod{m}
   \end{align*}
   \]

   By using Theorem 1.13 we can subtract pairwise those congruences and we obtain
   
   \[
   \begin{align*}
   x - x' & \equiv 0 \pmod{n} \\
   x - x' & \equiv 0 \pmod{m}
   \end{align*}
   \]

   Thus \( n | (x - x') \) and \( m | (x - x') \). Since \( \gcd(n, m) = 1 \), by Theorem 2.25 (or Theorem 1.42) we conclude that \( mn | (x - x') \). That is \( x \equiv x' \pmod{mn} \).

   **Comment:** The unique solution of Theorem 3.29 can be constructed as follows: \( x = mx_1 + nx_2 \) where \( x_1 \) and \( x_2 \) are solutions of the following congruences
   
   \[
   \begin{align*}
   x & \equiv mx_1 \equiv a \pmod{n} \\
   x & \equiv nx_2 \equiv b \pmod{m}
   \end{align*}
   \]

   These solutions \( x_1 \) and \( x_2 \) exist by Theorem 3.20 as \( \gcd(n, m) = 1 \) divides both \( a \) and \( b \).
3. Theorem 3.29 (Chinese Remainder Theorem).
Suppose \( n_1, n_2, \ldots, n_L \) are positive integers that are pairwise relatively prime, that is \( \gcd(n_i, n_j) = 1 \) for \( i \neq j, 1 \leq i, j \leq L \). Then the system of \( L \) congruences
\[
\begin{align*}
x \equiv a_1 & \pmod{n_1} \\
x \equiv a_2 & \pmod{n_2} \\
\vdots \\
x \equiv a_L & \pmod{n_L}
\end{align*}
\]
has a unique solution modulo the product \( n_1 n_2 \cdots n_L \).

Proof: We will construct a solution \( x \) of the form
\[
x = \frac{n_1 n_2 \cdots n_L}{n_1} x_1 + \frac{n_1 n_2 \cdots n_L}{n_2} x_2 + \ldots + \frac{n_1 n_2 \cdots n_L}{n_L} x_L
\]
Considering \( x \) modulo \( n_1, n_2, \ldots, n_L \) yields:
\[
\begin{align*}
\frac{n_1 n_2 \cdots n_L}{n_1} x_1 & \equiv a_1 \pmod{n_1} \\
\frac{n_1 n_2 \cdots n_L}{n_2} x_2 & \equiv a_2 \pmod{n_2} \\
\vdots \\
\frac{n_1 n_2 \cdots n_L}{n_L} x_L & \equiv a_L \pmod{n_L}
\end{align*}
\]
Because \( n_1, n_2, \ldots, n_L \) are pairwise relatively prime, no two of these integers have any primes in common in their prime factorizations. Thus \( (n_1 n_2 \cdots n_L)/n_1 = n_2 \cdots n_L \) has no primes in common with \( n_1 \) in their prime factorizations, and so \( \gcd(n_2 \cdots n_L, n_1) = 1 \). Therefore we can find a solution for \( x_1 \) in the first congruence above by Theorem 3.20. The same argument shows we can find desired values for \( x_2, \ldots, x_L \). Substituting for \( x_i \) into our expression for \( x \), we have found one solution to the original system.

Now consider any other solution \( x' \) to the same original system. By the same argument as in Theorem 3.28 we can prove that \( x - x' \) is a multiple of each \( n_1, n_2, \ldots, n_L \), and because these integers \( n_1 \) have no primes in common in their prime factorizations we can conclude \( x - x' \) is a multiple of \( n_1 n_2 \cdots n_L \). Therefore \( x \equiv x' \pmod{n_1 n_2 \cdots n_L} \).

4. Exercise 3.25.
In the pirates’ problem we need to find \( c \) (# of coins) such that
\[
\begin{align*}
c \equiv 3 & \pmod{17} \\
c \equiv 10 & \pmod{16} \\
c \equiv 0 & \pmod{15}
\end{align*}
\]
Using the construction of Theorem 3.29, we consider \( c = 16 \cdot 15 \cdot c_1 + 17 \cdot 15 \cdot c_2 + 17 \cdot 16 \cdot c_3 \), where
\[
16 \cdot 15 \cdot c_1 \equiv 3 \pmod{17} \quad 17 \cdot 15 \cdot c_2 \equiv 10 \pmod{16} \quad 17 \cdot 16 \cdot c_3 \equiv 0 \pmod{15}.
\]
To solve the above congruences, notice that they can be rewritten as
\[
\begin{align*}
(-1) \cdot (-2) \cdot c_1 & \equiv 3 \pmod{17} \quad (1) \cdot (-1) \cdot c_2 \equiv 10 \pmod{16} \quad (2) \cdot (1) \cdot c_3 \equiv 0 \pmod{15},
\end{align*}
\]
or
\[
\begin{align*}
2 \cdot c_1 & \equiv 3 \pmod{17} \quad c_2 \equiv -10 \pmod{16} \quad 2 \cdot c_3 \equiv 0 \pmod{15}.
\end{align*}
\]
The solutions are
\[
\begin{align*}
c_1 & \equiv 10 \pmod{17} \quad c_2 \equiv 6 \pmod{16} \quad c_3 \equiv 0 \pmod{15}.
\end{align*}
\]
Hence \( c = (16 \cdot 15 \cdot 10 + 17 \cdot 15 \cdot 6 + 17 \cdot 16 \cdot 0) = 3930 \equiv c' \pmod{4080} \).