Heron’s Formula for Triangular Area

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Heron of Alexandria

- Physicist, mathematician, and engineer
- Taught at the museum in Alexandria
- Interests were more practical (mechanics, engineering, measurement) than theoretical
- He is placed somewhere around 75 A.D. (±150)
Heron’s Works

- Automata
- Mechanica
- Dioptra
- Metrica
- Pneumatica
- Catoptrica
- Belopoeicia
- Geometrica
- Stereometrica
- Mensurae
- Cheirobalistra

The Aeolipile

Heron’s Aeolipile was the first recorded steam engine. It was taken as being a toy but could have possibly caused an industrial revolution 2000 years before the original.
Metrica

- Mathematicians knew of its existence for years but no traces of it existed
- In 1894 mathematical historian Paul Tannery found a fragment of it in a 13th century Parisian manuscript
- In 1896 R. Schöne found the complete manuscript in Constantinople.
- Proposition I.8 of Metrica gives the proof of his formula for the area of a triangle

How is Heron’s formula helpful?

How would you find the area of the given triangle using the most common area formula?

\[ A = \frac{1}{2} bh \]

Since no height is given, it becomes quite difficult...
Heron’s Formula

Heron’s formula allows us to find the area of a triangle when only the lengths of the three sides are given. His formula states:

\[ K = \sqrt{s(s-a)(s-b)(s-c)} \]

Where \(a\), \(b\), and \(c\), are the lengths of the sides and \(s\) is the semiperimeter of the triangle.

The Preliminaries...
**Proposition 1**

Proposition IV.4 of Euclid's Elements.

The bisectors of the angles of a triangle meet at a point that is the center of the triangles inscribed circle. (Note: this is called the incenter)

**Proposition 2**

Proposition VI.8 of Euclid's Elements.

In a right-angled triangle, if a perpendicular is drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle and to one another.
Proposition 3

In a right triangle, the midpoint of the hypotenuse is equidistant from the three vertices.

Proposition 4

If AHBO is a quadrilateral with diagonals AB and OH, then if $\angle HOB$ and $\angle HAB$ are right angles (as shown), then a circle can be drawn passing through the vertices A, O, B, and H.
Proposition 5

Proposition III.22 of Euclid’s Elements.
The opposite angles of a cyclic quadrilateral sum to two right angles.

Semiperimeter

The semiperimeter, s, of a triangle with sides a, b, and c, is

\[ s = \frac{a + b + c}{2} \]
Heron’s Proof...

The proof for this theorem is broken into three parts. Part A inscribes a circle within a triangle to get a relationship between the triangle’s area and semiperimeter. Part B uses the same circle inscribed within a triangle in Part A to find the terms s-a, s-b, and s-c in the diagram. Part C uses the same diagram with a quadrilateral and the results from Parts A and B to prove Heron’s theorem.
Restatement of Heron’s Formula

For a triangle having sides of length $a$, $b$, and $c$ and area $K$, we have

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s$ is the triangle’s semiperimeter.

Heron’s Proof: Part A

Let $ABC$ be an arbitrary triangle such that side $AB$ is at least as long as the other two sides.

Inscribe a circle with center $O$ and radius $r$ inside of the triangle.

Therefore, $OD = OE = OF$. 

![Diagram of Heron's Proof]

[Diagram showing the circle inscribed in the triangle with equal segments $OD = OE = OF$]
Heron’s Proof: Part A (cont.)

Now, the area for the three triangles \( \triangle AOB, \triangle BOC, \) and \( \triangle COA \) is found using the formula

\[
\text{Area } \triangle = \frac{1}{2} \text{(base)} \times \text{(height)}.
\]

Area \( \triangle AOB = \frac{1}{2} (AB)(OD) = \frac{1}{2} \cdot cr \)

Area \( \triangle BOC = \frac{1}{2} (BC)(OE) = \frac{1}{2} \cdot ar \)

Area \( \triangle COA = \frac{1}{2} (AC)(OF) = \frac{1}{2} \cdot br \)

We know the area of triangle ABC is \( K \). Therefore

\[
K = \text{Area}(\triangle ABC) = \text{Area}(\triangle AOB) + \text{Area}(\triangle BOC) + \text{Area}(\triangle COA)
\]

If the areas calculated for the triangles \( \triangle AOB, \triangle BOC, \) and \( \triangle COA \) found in the previous slides are substituted into this equation, then \( K \) is

\[
K = \frac{1}{2} \cdot cr + \frac{1}{2} \cdot ar + \frac{1}{2} \cdot br = r \left( \frac{a + b + c}{2} \right) = rs
\]
Heron’s Proof: Part B

When inscribing the circle inside the triangle ABC, three pairs of congruent triangles are formed (by Euclid’s Prop. I.26 AAS).

\[ \Delta AOD \cong \Delta AOF \]
\[ \Delta BOD \cong \Delta BOE \]
\[ \Delta COE \cong \Delta COF \]

Heron’s Proof: Part B (cont.)

- Using corresponding parts of similar triangles, the following relationships were found:

\[ AD = AF \]
\[ BD = BE \]
\[ CE = CF \]
\[ \angle AOD = \angle AOF \]
\[ \angle BOD = \angle BOE \]
\[ \angle COE = \angle COF \]
Heron’s Proof: Part B (cont.)

- The base of the triangle was extended to point G where AG = CE. Therefore, using construction and congruence of a triangle:

\[
BG = BD + AD + AG = BD + AD + CE
\]

\[
BG = \gamma(2BD + 2AD + 2CE)
\]

\[
BG = \gamma(2BD + AD + BE + CE + AF + CF)
\]

\[
BG = \gamma(AB + BC + AC)
\]

\[
BG = \gamma(c + a + b) = s
\]

Heron’s Proof: Part B (cont.)

- Since \(BG = s\), the semi-perimeter of the triangle is the long segment straighten out. Now, s-c, s-b, and s-a can be found.

\[
s - c = BG - AB = AG
\]

Since AD = AF and AG = CE = CF,

\[
s - b = BG - AC = \left( BD + AD + AG \right) - \left( AF + CF \right)
\]

\[
= \left( BD + AD + CE \right) - \left( AD + CE \right)
\]

\[
= BD
\]
Heron’s Proof: Part B (cont.)

Since $BD = BF$ and $AG = CE$,

$$s - a = BG - BC = (BD + AD + AG) - (BE + CE)$$
$$= (BD + AD + CE) - (BD + CE)$$
$$= AD$$

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Heron’s Proof: Part B (cont.)

- In Summary, the important things found from this section of the proof.

$$BG = \sqrt{2} \left( c + a + b \right) = s$$

$$s - c = AG$$

$$s - b = BD$$

$$s - a = AD$$
Heron’s Proof: Part C

- The same circle inscribed within a triangle is used except three lines are now extended from the diagram.
- The segment OL is drawn perpendicular to OB and cuts AB at point K.
- The segment AM is drawn from point A perpendicular to AB and intersects OL at point H.
- The last segment drawn is BH.
- The quadrilateral AHBO is formed.

Heron’s Proof: Part C (cont.)

- Proposition 4 says the quadrilateral AHBO is cyclic while Proposition 5 by Euclid says the sum of its opposite angles equals two right angles.

\[ \angle AHB + \angle AOB = 2 \text{ right angles} \]
Heron’s Proof: Part C (cont.)

- By congruence, the angles around the center O reduce to three pairs of equal angles to give:

  \[ 2\alpha + 2\beta + 2\gamma = 4 \text{ rt angles} \]

Therefore,

\[ \alpha + \beta + \gamma = 2 \text{ rt angles} \]

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Heron’s Proof: Part C (cont.)

- Since \( \beta + \alpha = \angle AOB \), and

\[ \alpha + \beta + \gamma = 2 \text{ rt angles} \]

\( \alpha + \angle AOB = 2 \text{ rt angles} = \angle AHB + \angle AOB \)

Therefore, \( \alpha = \angle AHB \).
Heron’s Proof: Part C (cont.)

- Since $\alpha = \angle AHB$ and both angles CFO and BAH are right angles, then the two triangles $\triangle COF$ and $\triangle BHA$ are similar.
- This leads to the following proportion using from Part B that $\frac{AG}{CF} = \frac{r}{OH}$:

$$\frac{AB}{AH} = \frac{CF}{OF} = \frac{AG}{r}$$

which is equivalent to the proportion

$$\frac{AB}{AG} = \frac{AH}{r} \quad (*)$$

Heron’s Proof: Part C (cont.)

- Since both angles KAH and KDO are right angles and vertical angles AKH and DKO are equal, the two triangles $\triangle KAH$ and $\triangle KDO$ are similar.
- This leads to the proportion:

$$\frac{AH}{AK} = \frac{OD}{KD} = \frac{r}{KD}$$

Which simplifies to

$$\frac{AH}{r} = \frac{AK}{KD} \quad (**$$)
Heron’s Proof: Part C (cont.)

- The two equations
  \[
  \frac{AB}{AG} = \frac{AH}{r} \quad (*) \quad \text{and} \quad \frac{AH}{r} = \frac{AK}{KD} \quad (**) 
  \]
  are combined to form the key equation:
  \[
  \frac{AB}{AG} = \frac{AK}{KD} \quad (***)
  \]

- By Proposition 2, \( \triangle KDO \) is similar to \( \triangle ODB \) where \( \triangle BOK \) has altitude \( OD=r \).

- This gives the equation:
  \[
  \frac{KD}{r} = \frac{r}{BD}
  \]
  which simplifies to
  \[
  (KD)(BD) = r^2 \quad (****)
  \]
  \( r \) is the mean proportional between magnitudes \( KD \) and \( BD \).
Heron’s Proof: Part C (cont.)

- One is added to equation (***)\(,\) the equation is simplified, then \(BG/BG\) is multiplied on the right and \(BD/BD\) is multiplied on the left, then simplified.

\[
\begin{align*}
\frac{AB}{AG} &= \frac{AK}{KD} \\
\frac{AB}{AG} + 1 &= \frac{AK}{KD} + 1 \\
\frac{AB + AG}{AG} &= \frac{AK + KD}{KD} \\
\frac{BG}{AG} &= \frac{AD}{KD}
\end{align*}
\]

Using the equation \(\frac{KD}{BD} = r^2\) (***) this simplifies to:

\[
\begin{align*}
\frac{BG}{AG} &= \frac{AD}{BD} \\
\frac{(BG)^2}{(AG)(BG)} &= \frac{(AD)(BD)}{r^2}
\end{align*}
\]

Heron’s Proof: Part C (cont.)

- Cross-multiplication of \(\frac{(BG)^2}{(AG)(BG)} = \frac{(AD)(BD)}{r^2}\) produced

\[
r^2(BG)^2 = (AG)(BG)(AD)(BD)
\]

Next, the results from Part B are needed. These are:

\[
\begin{align*}
BG &= s \\
s - b &= BD \\
sg - c &= AG \\
s - a &= AD
\end{align*}
\]
Heron’s Proof: Part C (cont.)

- The results from Part B are substituted into the equation:
  \[ r^2 \overline{BG}^2 = (AG)(BG)(AD)(BD) \]
  \[ r^2 s^2 = (s - c)(s)(s - b)(s - c) \]
- We know remember from Part A that \( K = rs \), so the equation becomes:
  \[ K = \sqrt{s(s - a)(s - b)(s - c)} \]
- Thus proving Heron’s Theorem of Triangular Area

Application of Heron’s Formula

We can now use Heron’s Formula to find the area of the previously given triangle

\[
s = \frac{1}{2}(17 + 25 + 26) = 34
\]

\[
K = \sqrt{34(34 - 17)(34 - 25)(34 - 26)} = \sqrt{41616} = 204
\]
Euler’s Proof of Heron’s Formula

Leonhard Euler provided a proof of Heron’s Formula in a 1748 paper entitled “Variae demonstrationes geometriae”

His proof is as follows...

Euler’s Proof (Picture)

For reference, this is a picture of the proof by Euler.
Euler’s Proof (cont.)

Begin with $\triangle ABC$ having sides $a$, $b$, and $c$ and angles $\alpha$, $\beta$ and $\gamma$

Inscribe a circle within the triangle

Let $O$ be the center of the inscribed circle with radius $r = OS = OU$

From the construction of the incenter, we know that segments $OA$, $OB$, and $OC$ bisect the angles of $\triangle ABC$ with $\angle OAB = \frac{\alpha}{2}$, $\angle OBA = \frac{\beta}{2}$, and $\angle OCA = \frac{\gamma}{2}$

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Euler’s Proof (cont.)

Extend $BO$ and construct a perpendicular from $A$ intersecting this extended line at $V$

Denote by $N$ the intersection of the extensions of segment $AV$ and radius $OS$

Because $\angle AOV$ is an exterior angle of $\angle AOB$, observe that

$$\angle AOV = \angle OAB + \angle OBA = \frac{\alpha}{2} + \frac{\beta}{2}$$
Euler’s Proof (cont.)

Because $\angle AOV$ is right, we know that $\angle AOV$ and $\angle OAV$ are complementary.

Thus, $\frac{\alpha}{2} + \frac{\beta}{2} + \angle OAV = 90^\circ$

But $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90^\circ$ as well.

Therefore, $\angle OAV = \frac{\gamma}{2} = \angle OCU$

Euler’s Proof (cont.)

Right triangles $\triangle OAV$ and $\triangle OCU$ are similar so we get $AV / VO = CU / OU = z / r$.

Also deduce that $\triangle NOV$ and $\triangle NAS$ are similar, as are $\triangle NAS$ and $\triangle BAV$, as well as $\triangle NOV$ and $\triangle BAV$.

Hence $AV / AB = OV / ON$.

This results in $\frac{z}{r} = \frac{AB}{ON} = \frac{x+y}{SN-r}$

So, $\frac{z}{SN} = r(x + y + z) = rs$.
Euler’s Proof (cont.)

Because they are vertical angles, $\angle BOS$ and $\angle VON$ are congruent, so

$$\angle OBS = 90^\circ - \angle BOS = 90^\circ - \angle VON = \angle ANS$$

$\Delta NAS$ and $\Delta BOS$ are similar

Hence, $\frac{SN}{AS} = \frac{BS}{OS}$

This results in $\frac{SN}{x} = \frac{y}{r}$

$$SN = \frac{(xy)}{r}$$

Euler’s Proof (cont.)

Lastly, Euler concluded that

$$Area(\Delta ABC) = rs = \sqrt{rs(rs)} = \sqrt{z(SN)(rs)}$$

$$= \sqrt{z\left(\frac{xy}{r}\right)rs} = \sqrt{sxyz} = \sqrt{s(s-a)(s-b)(s-c)}$$
Pythagorean Theorem

Heron’s Formula can be used as a proof of the Pythagorean Theorem

Pythagorean Theorem from Heron’s Formula

Suppose we have a right triangle with hypotenuse of length $a$, and legs of length $b$ and $c$

The semiperimeter is:

$$s = \frac{a+b+c}{2}$$
Pythagorean Thm. from Heron’s Formula (cont.)

\[
s - a = \frac{a+b+c}{2} - a = \frac{a+b+c}{2} - \frac{2a}{2} = \frac{-a+b+c}{2}
\]

Similarly

\[
s - b = \frac{a-b+c}{2} \quad \text{and} \quad s - c = \frac{a+b-c}{2}
\]

After applying algebra, we get...

Pythagorean Thm. from Heron’s Formula (cont.)

\[
(a + b + c)(-a + b + c)(a - b + c)(a + b - c)
\]
\[
= [(b + c) + a][(b + c) - a]a - (b - c)]a + (b - c)]
\]
\[
= [(b + c)^2 - a^2]a^2 - (b - c)^2
\]
\[
= a^2(b + c)^2 - (b + c)^2(b - c)^2 - a^4 + a^2(b - c)^2
\]
\[
= 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)
\]
Pythagorean Thm. from Heron’s Formula (cont.)

Returning to Heron’s Formula, we get the area of the triangle to be

\[ K = \sqrt{s(s - a)(s - b)(s - c)} \]

\[ = \sqrt{(\frac{a+b+c}{2})(\frac{-a+b+c}{2})(\frac{a-b+c}{2})(\frac{a+b-c}{2})} \]

\[ = \sqrt{\frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)}{16}} \]

Pythagorean Thm. from Heron’s Formula (cont.)

Because we know the height of this triangle is \( c \), we can equate our expression to the expression

\[ K = \frac{1}{2}bh = \frac{1}{2}bc \]

Equating both expressions of \( K \) and squaring both sides, we get

\[ \frac{b^2c^2}{4} = \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4)}{16} \]

Cross-multiplication gives us

\[ 4b^2c^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - (a^4 + b^4 + c^4) \]
Pythagorean Thm. from Heron’s Formula (cont.)

Taking all terms to the left side, we have

\[
\begin{align*}
(b^4 + 2b^2c^2 + c^4) - 2a^2b^2 - 2a^2c^2 + a^4 &= 0 \\
(b^2 + c^2)^2 - 2a^2(b^2 + c^2) + a^4 &= 0 \\
[ (b^2 + c^2) - a^2] &= 0 \\
(b^2 + c^2) - a^2 &= 0 \\
a^2 &= b^2 + c^2
\end{align*}
\]

Thus, Heron’s formula provides us with another proof of the Pythagorean Theorem.