The late 1600s and early 1700s was an exciting time period for mathematics. The subject flourished during this period. Math challenges were held among philosophers. The fundamentals of Calculus were created. Several geniuses made their mark on mathematics.
Gottfried Wilhelm Leibniz (1646-1716)

- Described as a universal genius by mastering several different areas of study.
- A child prodigy who studied under his father, a professor of moral philosophy.
- Taught himself Latin and Greek at a young age, while studying the array of books on his father’s shelves.
- At age 15 he entered the University of Leipzig, flying through his studies at such a pace that he completed his doctoral dissertation at Altdorf by 20.
Gottfried Wilhelm Leibniz

- He then began work for the Elector of Mainz, a small state when Germany divided, where he handled legal matters.
- In his spare time he designed a calculating machine that would multiply by repeated, rapid additions and divide by rapid subtractions.
- 1672-sent form Germany to Paris as a high level diplomat.
At this time his math training was limited to classical training and he needed a crash course in the current trends and directions it was taking to again master another area.

When in Paris he met the Dutch scientist named Christiaan Huygens.
Christiaan Huygens

- He had done extensive work on mathematical curves such as the “cycloid”.
- This is the path traced by a point fixed to the rim of a circle that is rolling along a horizontal path.
- His discoveries played a role in his design of the first successful pendulum clock.
- He built a reputation on his studies in physics and astronomy.
With such a wonderful resource, Leibniz was sure to succeed under the guidance of Huygens.

Huygens suggested for Leibniz to solve the determination of the sum of the reciprocals of the triangular numbers. More simply, numbers of the general form $k(k+1)/2$.

Soon he noticed a pattern in the numbers and was able to correctly evaluate the problem by manipulating the fractions.

Modern mathematicians voice certain reservations about the cavalier manipulations in this argument, but no one can deny the basic ingenuity of his approach.
This was just the beginning of Leibniz’s mathematical insights. He would soon be applying his great talents to questions Newton had addressed a decade earlier. By the time he had left Paris in 1676, he had discovered the fundamental principles of calculus. The four years spent in Paris saw him rise from novice to a giant in mathematics.
Meanwhile in England...

- Isaac Newton’s discoveries were known only to a select number of English mathematicians who had seen the handwriting manuscripts kept so quiet.
- 1673-On a visit to London Leibniz was one of the few to see some of these documents and was greatly impressed.
- Leibniz inquired further about the discoveries and Newton responded in two famous 1676 letters: epistola prior and the epistola posterior.
- Leibniz read intently.
The British cry, “FOUL!”

- 1684-Leibniz published his first paper on this amazing new mathematical method in the scholarly journal Acta Eruditorum, which he was the editor of, and his British counterparts cried “foul!”

- The paper was titled: “Novo Methodus pro Maximis et Minimis, itemque tangentibus, qua nec fractas, nec irrationals, quantitates moratur, et singulare pro illis calculi genus”--------In English: “A New Method for Maxima and Minima, as well as Tangents, which is impeded neither by Fractional nor Irrational Quantities, and a Remarkable Type of Calculus for this”.

The British cry, “FOUL!” (cont.)

- The title of this paper was where the subject took its name.
- Thus the world learned the calculus from Leibniz and not Newton.
- The British then accused him of plagiarism from his exchange of letters, visits to England, and the familiarity with the quietly circulating Newtonian manuscript.
Philosophers at War

- The bickering that followed does not constitute one of the more admirable chapters in the history of mathematics.
- At first they tried to stay out of it, but eventually all parties became involved.
- Leibniz freely admitted his contact with Newton but said the information given were only hints at results, not clear-cut methods.
- This form of calculus caught on quickly in Europe, meanwhile Newton still refused to publish anything on the subject.
Philosophers at War

- It was not until 1704 that Newton published a specific account of his method in the appendix to his *Opticks*.
- *De Analysi*, a more thorough account did not appear in print until 1711.
- Then a full-blown development of Newton’s ideas which was carefully written for learners, appeared only in 1736, nine years after his death.
- So tardy that Leibniz supporters returned the accusation of plagiarism to Newton himself.
The Dust Settles

- Both men deserve credit for independently developing virtually the same body of ideas.
- Not engaged in the controversy, Leibniz devoted his time to the variety of pursuits that had characterized his life.
- He was a major force in the creation of the Berlin Academy, bringing many great thinkers to Europe and putting Berlin on the intellectual map.
Leibniz died in 1716. His status had diminished and had never reveled in the same titanic reputation as Newton.

Where Sir Isaac Newton secluded himself from talented disciples, Leibniz had two of the most enthusiastic followers, Jakob and Johann Bernoullii of Switzerland.
The Bernoulli Family

- One of the most distinguished families in mathematical and scientific history.
- Many mathematical and scientific accomplishments were achieved over several generations of Bernoullis.
- This was a family filled with jealousy and rivalry.
Family Reunion (cont…)

- Jakob (1654-1705) and Johann (1667-1748), sons of Nicolaus.
- These brothers are the first Bernoullis to make an impact in mathematical history.
- There existed a bitter sibling rivalry between these two.
- Were among the first to realize the power of calculus.
Jakob Bernoulli

- Was to study philosophy and theology by his parents wishes, but wanted to study mathematics instead, this would be a trend for all the great Bernoullis.
- Graduated with a masters degree in philosophy, but soon turned to mathematics.
- He and his brother Johann after him, were greatly influenced by Leibniz, among others.
Jakob Bernoulli (cont...)  

- From 1687 until death, held Chair of Mathematics at Basel University. 
- Contributed to the use of Polar Coordinates, discovery of so-called isochrone, and proposed the problem of isoperimetric figures among others. 
- Wrote a break through book in probability called Ars conjectandi.
Jakob Bernoulli (cont...)

- Jakob is famous for...
- The Bernoulli distribution and Bernoulli theorem of statistics and probability.
- The Bernoulli numbers and Bernoulli polynomials of number-theory interest.
- The lemniscate of Bernoulli.
  \[ r^2 = a \cos(2\theta) \]
- First used the word integral in a calculus sense in his solution of the isochrone.
Johann Bernoulli

- Was an even bigger contributor to mathematics than Jakob.
- Very jealous and cantankerous
- Jakob taught Johann all he knew.
- A friendly correspondence between the two turn into a bitter rivalry once Johann matched and surpassed Jakob in mathematical knowledge.
- Johann was always jealous of Jakob because he had achieved so much and try very hard to become better.
Johann Bernoulli (cont...)

- Johann was an excellent teacher and exposed the power of calculus to continental Europe.
- Solved a problem his brother posed, problem of the catenary, became his first independent work.
- Later he taught l’Hospital new methods of calculus in exchange for large sums of money.
- L’Hospital’s rule, published in L’Hospital’s book, was really Johann’s solution sent to L’Hospital in a lesson.
- This upset Johann tremendously, but no one believed him when he protested, a proof was later found in 1922 in Basel.
- Johann took over Mathematics Chair at Basel when Jakob died.
Johann Bernoulli (cont...)

- Johann also worked on the problem of the Brachistochrone and tautochrone.
- He proposed the Brachistochrone problem to the entire mathematics world in 1696.
- Not to be outdone Jakob then proposed the isoperimetric problem (minimizing the area enclosed by a curve).
- Johann also made contributions to mechanics with his work in kinetic energy.
Other famous Bernoullis

- Nicolaus(2), son of Nicolaus, brother to Jakob and Johann, mathematician of lesser fame.
- Nicolaus(3), son of Nicolaus(2), nephew of Jakob and Johann.
- Nicolaus(4)(1695-1726), Daniel(1700-1782), and Johann II(1710-1790), sons of Johann.
Other famous Bernoullis

- Nicolaus(4) was a very promising mathematician that proposed the St. Petersburg Paradox.
- He tragically died in St. Petersburg 8 months after arriving and was succeeded by Daniel, his younger brother.
Other famous Bernoullis

- Daniel was the most famous of Johann’s sons.
- Johann and Daniel both entered a scientific contest at the University of Paris and they tied.
- Johann was furious he was compared to his offspring and banned him from his house.
- Daniel later published Hydrodynamica, which Johann tried to steal and rename Hydraulica.
- Johann carried a grudge against his son until his death.
Other Famous Bernoullis

- Johann arranged for Leonard Euler to be sent to St. Petersburg to study with Daniel.
- Euler and Daniel work with mechanics and the study of flexible and elastic bodies.
- Daniel’s most prolific work, Hydrodynamica, was based on a single principle, the conservation of energy.
- Developed Bernoulli’s equation, a key to describing fluids in motion and fluid dynamics.
The Challenge of the Brachistochrone

- Points A and B at different heights above ground
- Infinite curves joining A and B
- Imagine object traveling down curve AMB in the shortest time
- Brachistochrone: Greek for “shortest” and “time”
Solution

- Initial guess is straight line because shortest distance
- Johann proposed problem to the mathematical world
- On deadline set by Johann, he received solutions from:
  - Leibniz
  - Jakob
  - Marquis de l’Hospital
  - Anonymous from England (Isaac Newton)
Great Theorem: The Divergence of the Harmonic Series

- Concocted by Johann Bernoulli, but appeared in brother Jakob's book
- Harmonic Series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \text{is infinite} \]
Johann’s Proof

\[ A = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{k} + \ldots \]

\[ A = \frac{1}{2} + \frac{2}{6} + \frac{3}{12} + \frac{4}{20} + \frac{5}{30} + \ldots \]
Johann’s Proof

\[
C = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = 1
\]

\[
D = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = C - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}
\]

\[
E = \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots = D - \frac{1}{6} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}
\]

\[
F = \frac{1}{20} + \frac{1}{30} + \ldots = E - \frac{1}{12} = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}
\]

\[
G = \frac{1}{30} + \ldots = F - \frac{1}{20} = \frac{1}{4} - \frac{1}{20} = \frac{1}{5}
\]
Johann’s Proof

Sum down two leftmost columns:

\[ C + D + E + F + ... = \frac{1}{2} + \left( \frac{1}{6} \right) + \left( \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \right) + \left( \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} \right) + ... = A \]

Sum down leftmost and rightmost columns:

\[ C + D + E + F + G + ... = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... = 1 + A \]
Therefore...

\[ C + D + E + F + G + ... = A = 1 + A \]

“The whole equals the part” so A must be infinite
Previous Proofs of Harmonic Series Divergence

- Earliest-Nicole Oresme (1323-1382)
- Verbal

“...add to a magnitude of 1 foot: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ foot, etc.; the sum of which is infinite. In fact, it is possible to form an infinite number of groups of terms with a sum greater than $\frac{1}{2}$. Thus: $\frac{1}{3} + \frac{1}{4}$ is greater than $\frac{1}{2}$; $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ is greater than $\frac{1}{2}$; $\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}$ is greater than $\frac{1}{2}$, etc.”
Replace groups of fractions in the harmonic series by smaller fractions that sum to $\frac{1}{2}$

$$1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1$$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > \frac{3}{2} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = \frac{4}{2}$$

$$1 + \frac{1}{2} + \ldots + \frac{1}{8} + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) > \frac{4}{2} + \left(\frac{1}{16} + \ldots + \frac{1}{16}\right) = \frac{5}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k} > \frac{k+1}{2}$$
Oresme’s Proof

For $k = 9$

\[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{512} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^9} > \frac{9 + 1}{2} = 5 \]

For $k = 99$

\[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^9} > \frac{99 + 1}{2} = 50 \]

For $k = 9999$

\[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^{9999}} > \frac{9999 + 1}{2} = 5000 \]

Therefore, harmonic series exceeds any finite quantity.
Previous Proofs of Harmonic Series Divergence

- Italian Pietro Mengoli (1625-1686)
- Proof dates back to 1647
Mengoli’s Proof

• Preliminary Result:

If $a > 1$, then

\[
\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1} > \frac{3}{a}
\]

• Proof:

\[
2a^3 > 2a^3 - 2a = 2a(a^2 - 1)
\]

\[
\frac{2a^3}{a^2(a^2 - 1)} > \frac{2a(a^2 - 1)}{a^2(a^2 - 1)}
\]

\[
\frac{2a}{a^2 - 1} > \frac{2}{a}
\]
Mengoli’s Proof

\[
\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1} = \frac{1}{a} + \left( \frac{1}{a-1} + \frac{1}{a+1} \right)
\]

\[
= \frac{1}{a} + \frac{2a}{a^2 - 1}
\]

\[
> \frac{1}{a} + \frac{2}{a}
\]

\[
= \frac{3}{a}
\]

by a bit of algebra

by the inequality above
Mengoli’s Proof

\[ H = 1 + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \left( \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) + \left( \frac{1}{11} + \frac{1}{12} + \frac{1}{13} \right) + \ldots \]

\[ > 1 + \left( \frac{3}{3} \right) + \left( \frac{3}{6} + \frac{3}{9} \right) + \left( \frac{3}{12} + \frac{3}{15} \right) + \ldots \]

\[ = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots \]

\[ = 2 + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \left( \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) + \left( \frac{1}{11} + \frac{1}{12} + \frac{1}{13} \right) + \ldots \]

\[ > 2 + \left( \frac{3}{3} \right) + \left( \frac{3}{6} + \frac{3}{9} \right) + \left( \frac{3}{12} + \frac{3}{15} \right) + \ldots \]

\[ = 2 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots \]

\[ = 3 + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right) + \left( \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) + \left( \frac{1}{11} + \frac{1}{12} + \frac{1}{13} \right) + \ldots \]

and so on.