2. (a) Mr. Akiku has 32 18-cent and 29-cent stamps all told. The stamps are worth $8.07. How many of each kind of stamp does he have? (b) Summarize your solution method in one or two carefully written sentences.  
3. Make up a problem similar to problems 1 and 2.  
4. (a) Place the digits 4, 6, 7, 8, and 9 in the circles to make the sums across and vertically equal 19. 

(b) Is there more than one answer to part (a)? Explain briefly. Digits should be used once and only once. There are no different solutions other than interchanging the 6 and 9 or the 7 and 8 or rotating the figure.  
5. Xin has nine coins with a total value of 48 cents. What coins does Xin have?  
6. Who am I? If you multiply me by 5 and subtract 8, the result is 52.  
7. Who am I? If you multiply me by 15 and add 28, the result is 103.  
8. Make up a problem like problems 5 and 6.  
9. (a) Using each of 1, 2, 3, 4, 5, and 6 once and only once, fill in the circles so that the sums of the numbers on each of the three sides of the triangle are equal.  

(b) Does part (a) have more than one solution?  
(c) Write up a brief but careful description of the thought process you used in solving this problem.  
10. In this diagram, the sum of any two horizontally adjacent numbers is the number immediately below and between them. Using the same rule of formation, complete these arrays.
1.1 Continued

If possible, complete each of these diagrams so that the same pattern holds.

(a) \[ \begin{array}{ccc} \bigcirc & \bigcirc & \bigcirc \\ 15 & 11 & 20 \\ \bigcirc & \bigcirc & \bigcirc \end{array} \]
(b) \[ \begin{array}{ccc} \bigcirc & \bigcirc & \bigcirc \\ 12 & 10 & 21 \\ \bigcirc & \bigcirc & \bigcirc \end{array} \]

(c) \[ \begin{array}{ccc} \bigcirc & \bigcirc & \bigcirc \\ 16 & 9 & 16 \\ \bigcirc & \bigcirc & \bigcirc \end{array} \]
(d) \[ \begin{array}{ccc} \bigcirc & \bigcirc & \bigcirc \\ 13 & 10 & 8 \\ \bigcirc & \bigcirc & \bigcirc \end{array} \]

12. Study this sequence of numbers: 3, 4, 7, 11, 18, 29, 47, 76. Note that 3 + 4 = 7, 4 + 7 = 11, 7 + 11 = 18, and so on. Use the same rule to complete these sequences.

(a) 1, 2, 3, __________, __________
(b) 2, __________, 8, __________, __________
(c) 3, __________, 13, __________, __________
(d) 2, __________, __________, 26
(e) 2, __________, __________, 11

13. (a) Use each of the numbers 2, 3, 4, 5, and 6 once and only once to fill in the circles so that the sum of the numbers in the three horizontal circles equals the sum of the numbers in the three vertical circles.

(b) Can you find more than one solution?
(c) Can you have a solution with 3 in the middle of the top row? Explain in two carefully written sentences.

14. (a) This is a magic square. Compute the sums of the numbers in each row, column, and diagonal of the square and write your answers in the appropriate circles. Each sum is 15.

(b) Interchange the 2 and 8 and the 4 and 6 in the array in part (a) to create this magic subtraction square. For each row, column, and diagonal, add the two end entries and subtract the middle entry from this sum. Each result is 5.

15. (a) Write the digits 0, 1, 2, 3, 4, 5, 6, 7, and 8 in the small squares to create another magic square. (Hint: Relate this to problem 14. Also, you may want to write these digits on nine small squares of paper that you can move around easily to check various possibilities.)

(b) Make a magic subtraction square using the numbers 0, 1, 2, 3, 4, 5, 6, 7, and 8. Digits should be used once and only once.

16. Make up a guess and check problem of your own and solve it.
Look Back

Like the problem of the pigs and chickens on Old MacDonald’s farm, this problem may not immediately suggest drawing a picture. However, having seen pictures used to solve these problems will help you to see how pictures can be used to solve other, even vaguely related, problems.

Problem Set 1.2

1. Diedre is thinking of a number. If you multiply it by 5 and add 13, you get 48. Could Diedre’s number be 10? Why or why not?

2. Lisa is thinking of a number. If you multiply it by 7 and subtract 4, you get 17. What is the number?

3. Jicky is thinking of a number. Twice the number increased by 1 is 5 less than 3 times the number. What is the number? (Hint: For each guess, compute two numbers and compare.)

4. In Mrs. Garcia’s class they sometimes play a game called Guess My Rule. The student who is It makes up a rule for changing one number into another. The other students then call out numbers and the person who is It tells what number the rule gives back. The first person in the class to guess the rule then becomes It and gets to make up a new rule.

(a) For Juan’s rule the results were

Numbers chosen | 2 | 5 | 4 | 0 | 8
---|---|---|---|---|---
Numbers Juan gave back | 7 | 22 | 17 | -3 | 37

Could Juan’s rule have been, “Multiply the chosen number by 5 and subtract 3”? Could it have been, “Reduce the chosen number by 1, multiply the result by 5, and then add 2”? Are these rules really different? Discuss briefly.

(b) For Mary’s rule, the results were

Numbers chosen | 3 | 7 | 1 | 0 | 9
---|---|---|---|---|---
Numbers Mary gave back | 10 | 50 | 2 | 1 | 82

What is Mary’s rule?

(c) For Peter’s rule, the results were

Numbers chosen | 0 | 1 | 2 | 3 | 4
---|---|---|---|---|---
Numbers Peter gave back | 7 | 10 | 13 | 16 | 19

Observe that the students began to choose the numbers in order starting with 0. Why is that a good idea? What is Peter’s rule?

5. As in Example 1.3, the numbers in the big circles are the sums of the numbers in the two small adjacent circles. Place numbers in the empty circles in each of these arrays so that the same scheme holds.

(a) 

(b) 

(c) 

(d)

6. How many different amounts of money can you pay if you use four coins including only nickels, dimes, and quarters?
7. How many different ways can you make change for a 50-cent coin using quarters, nickels, dimes, and pennies?

8. List the three-digit numbers that can be written using each of the digits 2, 5, and 8 once and only once.

9. List the three-digit numbers that can be written using each of 0, 3, and 5 once and only once.

10. When Anita made a purchase she gave the clerk a dollar and received 21 cents in change. Complete this table to show what Anita’s change could have been.

<table>
<thead>
<tr>
<th>Number of Dimes</th>
<th>Number of Nickels</th>
<th>Number of Pennies</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

11. Julie has 25 pearls. She put them in three velvet bags with an odd number of pearls in each bag. What are the possibilities?

12. A rectangle has an area of 120 cm². Its length and width are whole numbers.
   (a) What are the possibilities for the two numbers?
   (b) Which possibility gives the smallest perimeter?

13. The product of two whole numbers is 96 and their sum is less than 30. What are the possibilities for the two numbers?

14. Peter and Jill each worked a different number of days, but earned the same amount of money. Use these clues to determine how many days each worked:
   - Peter earned $20 a day.
   - Jill earned $30 a day.
   - Peter worked five more days than Jill.

15. Bob can cut through a log in one minute. How long will it take Bob to cut a 20-foot log into 2-foot sections? (Hint: Draw a diagram.)

16. How many posts does it take to support a straight fence 200 feet long if a post is placed every 20 feet?

17. How many posts does it take to support a fence around a square field measuring 200 feet on a side if posts are placed every 20 feet?

18. Albright, Badgett, Chalmers, Dawkins, and Ertl all entered the primary to seek election to the city council. Albright received 2000 more votes than Badgett and 4000 fewer than Chalmers. Ertl received 2000 votes fewer than Dawkins and 5000 votes more than Badgett. In what order did each person finish in the balloting?

19. Nine square tiles are laid out on a table so that they make a solid pattern. Each tile must touch at least one other tile along an entire edge. The squares all have sides of length one.

(a) What are the possible perimeters of the figures that can be formed? (The perimeter is the distance around the figure.)
(b) Which figure has the least perimeter?

20. For each of the strategies below, write a word problem that would use the method. Show the solution you have in mind.
   (a) Guess and Check
   (b) Make an orderly list
   (c) Draw a diagram

Teaching Concepts

21. Read one of the following from NCTM’s Principles and Standards for School Mathematics:
   (a) Problem Solving Standard for Grades Pre-K–2, pages 116–121
   (b) Problem Solving Standard for Grades 3–5, pages 182–187
   (c) Problem Solving Standard for Grades 6–8, pages 256–261

Write a critique of the standard you read, emphasizing your own reaction. How do the recommendations compare with your own school experience?

Thinking Cooperatively

22. There is a 2-mile long traffic jam on the highway. How many cars are in the traffic jam? (Hint: This problem is purposefully vague and quite open-ended. It has been used successfully with fourth grade students in Germany.) Discuss in your groups what information is needed to come up with a solution.
the elements in the nth row of the triangle is $2^n$. In an attempt to argue that this guess was correct, we considered the special case of obtaining the fourth row from the third. This showed that the sum of the elements in the fourth row was twice the sum of the elements in the third row. Since the argument did not depend on the actual numbers appearing in the third and fourth rows, but only on the general rule of formation of the triangle, it would hold for any two consecutive rows and so actually proves that our conjecture is correct. This type of argument is called **arguing from a special case** and is an important problem-solving strategy.

### Problem Set 1.3

1. Look for a pattern and fill in the next three blanks with the most likely choices for each sequence.
   (a) 2, 5, 8, 11, ___, ___, ___, 17, ___, 20
   (b) −5, −3, −1, 1, ___, ___, ___, 5, 7
   (c) 1, 1, 3, 3, 6, 6, 10, ___, 15, 15
   (d) 1, 3, 4, 7, 11, ___, 29, 47
   (e) 2, 6, 18, 54, 162, ___, 486, 1458

2. (a) Fill in the blanks to continue this dot sequence in the most likely way.


   (b) What number sequence corresponds to the sequence of dot patterns of part (a)?

3. Sequences like 2, 5, 8, …, where each term is greater (or less) than its predecessor by a constant amount, are called **arithmetic** (a-rith-me-tic) **progressions**. Find the number of terms in each of these arithmetic progressions.
   (a) 5, 7, 9, …, 35
   (b) −4, 1, 6, …, 46
   (c) 3, 7, 11, …, 67

4. Consider the sequence 8, 5, 2, −1, …, −52.
   (a) Is the sequence an arithmetic progression? Why or why not?
   (b) How many terms are there in the sequence?

5. (a) Fill in the blanks to continue this sequence of equations.

   \[
   \begin{align*}
   1 & = 1 \\
   1 + 2 + 1 & = 4 \\
   1 + 2 + 3 + 2 + 1 & = 9 \\
   1 + 2 + 3 + 4 + 3 + 2 + 1 & = 16 \\
   \frac{1 + 2 + 3 + 4 + 3 + 2 + 1}{1 + 2 + 3 + 4 + 3 + 2 + 1} & = \frac{1 + 2 + 3 + 4 + 3 + 2 + 1}{1 + 2 + 3 + 4 + 3 + 2 + 1} \\
   (b) & \text{ Compute this sum. The sum is } 100^2 = 10,000.
   
   1 + 2 + 3 + \cdots + 99 + 100 + 99 + \cdots + 3 + 2 + 1 = \_
   
   (c) & \text{ Fill in the blank to complete this equation.}
   
   1 + 2 + 3 + \cdots + (n - 1) + n + (n - 1) + \cdots + 3 + 2 + 1 = \_

6. (a) Fill in the blanks to continue this sequence of equations.

   \[
   \begin{align*}
   1 & = 0 + 1 \\
   1 + 3 + 1 & = 1 + 4 \\
   1 + 3 + 5 + 3 + 1 & = 4 + 9 \\
   \frac{1 + 3 + 5 + 3 + 1}{1 + 3 + 5 + 3 + 1} & = \frac{1 + 3 + 5 + 3 + 1}{1 + 3 + 5 + 3 + 1} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} & = \frac{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)}{1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)} \\
   (b) & \text{ What expression, suggested by part (a), should be placed in the blank to complete this equation? (n)}
   
   1 + 3 + 5 + \cdots + (2n - 3) + (2n - 1)
   + (2n - 3) + \cdots + 5 + 3 + 1 = \_

7. Writers of standardized tests often pose questions like "What is the next term in the sequence 2, 4, 8, …?"
   (a) How would you answer this question?
   (b) Evaluate the expressions $2^n$, $n^2 - n + 2$, and $n^3 - 5n^2 + 10n - 4$ in the following chart by replacing $n$ successively by 1, 2, 3, and 4.
(c) In light of the results in (b), what criticism would you make of the test writer who would write a test question like that above? Compare the wording above with that in problem 1 of this problem set.

8. Here is the start of a 100 chart.

\[
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}
\]

Shown below are parts of the chart. Without extending the chart, determine which numbers should go in the lavender squares.

9. Five blue (B) and five red (R) discs are lined up in the B B B B R R R R R arrangement shown.

Switching just two adjacent discs at a time, what is the least number of moves you can make to achieve the B R B R B R B R B R arrangement shown here?

(Hint: How many moves are required to rearrange B B R R to B R B R? B B B B R R to B B B B B R? and so on.)

10. (a) Complete the next two of this sequence of equations.

\[
\begin{align*}
1 &= 1 \\
1 - 4 &= -3 \\
1 - 4 + 9 &= 6 \\
1 - 4 + 9 - 16 &= -10 \\
\end{align*}
\]

(b) Write the seventh and eighth equations in the sequence of equations of part (a). (Hint: Have you encountered the number sequence 1, 3, 6, 10, \ldots before?)

(c) Write general equations suggested by parts (a) and (b) for even \( n \) and for odd \( n \), where \( n \) is the number of the equation.

11. (a) How many rectangles are there in each of these figures? (Note: Rectangles may measure 1 by 1, 1 by 2, 1 by 3, and so on.)

\[
\begin{align*}
\text{a} & \quad \text{1 rectangle} \\
\text{a} \quad \text{b} & \quad \text{3 rectangles} \\
\text{a} \quad \text{b} \quad \text{c} & \\
\text{a} \quad \text{b} \quad \text{c} \quad \text{d} & \\
\end{align*}
\]
(b) How many rectangles are in this figure? 

\[
\begin{array}{cccccccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \text{g} & \text{h} & \text{i} & \text{j} & \text{k} & \text{l} & \text{m} & \text{n} & \text{o} & \text{p}
\end{array}
\]

(c) How many rectangles are in a $1 \times n$ strip like that in part (b)?

(d) Argue that your guess in part (c) is correct, giving a lucid and careful write-up. (Hint: How many of each type of rectangle begin with each small square?)

12. If one must always move downward along the lines of the grid shown, how many different paths are there from point $A$ to each of these points?

13. If one must always move upward or to the right on each of the grids shown, how many paths are there from $A$ to $B$?

14. If one must follow along the paths of the following diagram in the direction of the arrows, how many paths are there from $C$ to $E$?

15. How many chords are determined by joining dots on a circle if there are 

(a) 4 dots?  
(b) 10 dots?  
(c) 100 dots?  
(d) $n$ dots?

(e) Argue that your solution to part (d) is correct.

16. (a) How many games are played in a round-robin tournament with 10 teams if every team plays every other team once?  

(b) How many games are played if there are 11 teams?  

(c) Is this problem related to problem 15 of this problem set? If so, how?

17. Here is an addition table.

\[
\begin{array}{ccccccccccc}
\text{+} & \text{0} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} \\
\hline
\text{0} & \text{0} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} \\
\text{1} & \text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} \\
\text{2} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} \\
\text{3} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} \\
\text{4} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} \\
\text{5} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} \\
\text{6} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} \\
\text{7} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16} \\
\text{8} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16} & \text{17} \\
\text{9} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16} & \text{17} & \text{18}
\end{array}
\]

(a) Find the sum of the entries in these squares of entries from the addition table.

\[
\begin{array}{cccc}
2 & 3 & 5 & 6 \\
3 & 4 & 6 & 7 \\
11 & 12 & 13 & 14 \\
35 & 36 & 37 & 38
\end{array}
\]
What is the final answer after all nine steps? The result at step 9 is 10,000y + z. However, 10,000y takes the three-digit number y (257 in our example) and adds four zeros (to get 2,570,000). Thus, the first three digits of the seven-digit number 10,000y + z are the digits of the number y.

What are the last four digits of 10,000y + z? They are exactly the digits of z (for example, 2,570,000 + 6821 = 2,576,821). The "math trick" works because 10,000y + z is a way to express in algebra the phone number (without area code) y-z (that's y dash z, not subtraction). Note that we didn't need a calculator to do the problem!

Look Back

In the explanation of Amanda's phone number problem, we used "algebra as the great explainer" because we were trying to show that something involving numbers was always correct. At first, we thought of solving the problem using only one variable but recognized that there were really two different numbers involved; that is, a phone number is made up of two parts (plus an area code) and we needed different names for the two of them. We then went through some algebraic steps and finished the problem. Since we proved the result for all seven-digit numbers, not just some specific ones, we have given a carefully reasoned argument that shows the result is always true. This approach is also an example of the "Proof and Reasoning Standard" of the NCTM.

Problem Set 1.4

Understanding Concepts

1. (a) Draw three diagrams to continue this dot sequence. [Image]

(b) What number sequence corresponds to the pattern of part "_"?

(c) What is the tenth term in the sequence of part (b)? the one-hundredth term? .

(d) Which even number is 2n? T

(e) What term in the sequence is 2402?

2. Example 1.1 (Guessing Toni's Number on page 5 of Section 1.1) was solved using guess and check. Do the example again applying algebra this time. (Suggestion: Use a variable.)

3. Jackson is thinking of a number, which if you triple it and subtract 13, ends with a result of 2. What number is Jackson thinking of? (Suggestion: Use a variable.)

4. Maria picks an integer and divides it by two and then adds 12 to what she has. When she tells you that she now has 10, can you tell her what integer she started with? (Suggestion: Use a variable.)

5. The first three trains in a sequence of trapezoid trains are shown below. Verify that the number of toothpicks required to form a train with 6 trapezoidal cars is given by the formula \( t = 1 + 4c \).

6. A rectangular table seats 6 people, 1 person on each end and 2 on each of the longer sides. Thus, two tables placed end to end seat 10 people.

   (a) How many people can be seated if \( n \) tables are placed end to end?

   (b) How many tables, set end to end, are required to seat 24 people?

7. (Handshake Problem) There are \( n \) people in the room and each of them will shake hands with every other person once and only once. The general question of how many handshakes take place is best done through the insights gained by experimenting (Problem-Solving Strategy 6).

   (a) If there are 3 people in the room, how many handshakes are made?

   (b) If there are 6 people, how many handshakes are made?

   (c) If there are 200 people, how many handshakes are made?

   (d) If there are \( n \) people, how many handshakes are made?
1.4 Algebra as a Problem-Solving Strategy

Imagination can generate many variants on the Handshake Problem. This exercise gives rise through the use of the Mathematical Habit of the Mind to a number of other problems.

8. (Feuding Handshake Problem) There are 100 people in the room, half of whom don’t speak to the other half. Assume that if they won’t speak to each other, they won’t shake hands. How many handshakes are there if everyone shakes hands only once (if they shake at all)?

9. (Marital Handshake Problem) There are 100 people consisting of 50 married couples in a room. Assuming that no husband or wife shake each other’s hand but everyone else shakes hands exactly once, how many handshakes are made?

10. (Your Handshake Problem) Make up a handshake problem similar to the three above.

11. Old MacDonald has 100 chickens and 100 goats in the barnyard. Altogether, there are 286 feet. How many chickens and how many goats are in the barnyard? (Suggestion: Use Problem-Solving Strategy 7.)

12. Consider the following sequence of equations:

\[ \begin{align*}
1 & = 1 \\
3 + 5 & = 8 \\
7 + 9 + 11 & = 27 \\
13 + 15 + 17 + 19 & = 64
\end{align*} \]

(a) Fill in the blanks to continue the sequence of equations.

(b) Guess a formula for the number on the right of the nth equation.

(c) Check that the expression \( n^2 - n + 1 \) generates the first number in the sum on the left of each equation and that \( n^2 + n - 1 \) generates the last number in these sums.

(d) Use the result of part (c) to prove that your guess to part (b) is correct. (Hint: How many terms are in the sum on the left of the nth equation?)

13. We have already considered the triangular numbers,

\[ T_n = \frac{n(n+1)}{2} \]

and the square numbers,

\[ S_n = n^2 \]

(a) Draw the next two figures to continue this sequence of dot patterns.

(b) List the sequence of numbers that corresponds to the sequence of part (a). These are called pentagonal numbers.

(c) Complete this list of equations suggested by parts (a) and (b).

\[ \begin{align*}
1 & = 1 \\
1 + 4 & = 5 \\
1 + 4 + 7 & = 12 \\
1 + 4 + 7 + 10 & = 22 \\
\vdots & = \vdots \\
\vdots & = \vdots
\end{align*} \]

Observe that each pentagonal number is the sum of an arithmetic progression.

(d) Compute the tenth term in the arithmetic progression 1, 4, 7, 10, \ldots .

(e) Compute the tenth pentagonal number.

(f) Determine the nth term in the arithmetic progression 1, 4, 7, 10, \ldots .

(g) Compute the nth pentagonal number, \( p_n \).

14. (a) The hexagonal numbers are associated with this sequence of dot patterns. Complete the next two diagrams in the sequence.

(b) Write the first five hexagonal numbers.

(c) What is the tenth hexagonal number?

(d) Compute a formula for \( h_n \), the nth hexagonal number.

15. Example 1.3 on page 11 of Section 1.2 was solved using guess and check. Do the example again but this time with algebra. (Use three variables.)

16. The figure below shows an addition pyramid, in which each number is the sum of the two numbers immediately below.
25. In Section 1.1, Mr. Capek’s fifth grade class found multiple methods of solving “Old MacDonald’s problem.” How would you solve it using algebra?

26. How would you generalize the formula of exercise 20?
   (a) If x and y differ by 2, show that the difference of their squares is twice their sum.
   (b) To formulate a generalization of part (a), what condition on the difference between x and y would you suggest? Write a problem which generalizes (a) and prove your result using algebra.

27. Algebra can be used to prove results, as we will see in Chapter 4. While the intent of this text is not to emphasize formal proofs, we do want to show the power of algebra. Here’s a beginning problem: Show that the square of an even integer is a multiple of 4. (Hint: Start with the definition! We know that if x is an even number, then there is another integer n with x = 2n. What is x^2?)

From State Student Assessments

28. (Illinois, Grade 5)
   \[ \bigstar + \bigstar = 10 \]
   \[ \bigstar - \bigstar = 2 \]
   Each big star has the same value.
   Each big star has the same value.
   What is the value of the big star?

29. (Kentucky, Grade 5)
   José had 64 baseball cards. He gave 12 cards to his sister. Then he divided the remaining cards equally among his four friends. How many cards did each of his friends get?
   (a) 13 cards
   (b) 16 cards
   (c) 17 cards
   (d) 18 cards

30. (Kentucky, Grade 8).
   For this open-response question, use the grid provided in this test booklet to create any required charts or graphs.

   Two small children were playing a game. The goal of the game was to be the first one to reach the door. The children started the game by standing 20 feet away from the door, and then they each took a turn to do the following:
   • Child A moved one-half the distance between herself and the door on each move.
   • Child B moved 1 foot toward the door on each move.

   (a) How far was each child from the door after the first move? A is 10 feet away and B is 19 feet away
   (b) After four moves, which child was closer to the door? Show your work.
   (c) Child A claimed that the game was unfair because she would never reach the door. Explain why her statement is correct or incorrect.

   Note: This is the exact problem from the Kentucky, Grade 8, exam. There is no test booklet contained in this text.

1.5 Additional Problem-Solving Strategies

Strategies

- Work backward.
- Eliminate possibilities.
- Use the pigeonhole principle.

**PROBLEM-SOLVING STRATEGY 8: Work Backward**

Start from the desired result and work backward step-by-step until the initial conditions of the problem are achieved.

Many problems require that a sequence of events occurs that results in a desired final outcome. These problems at first seem obscure and intractable, and you may be tempted to try a guess and check approach. However, it is often easier to work backward from the end result to see how the process would have to start to achieve the desired end. To make this more clear, consider the following example.