Here is a set of review problems.

1. Find an equation of the tangent plane to the surface \( x^2 + z^2e^{y-x} = 13 \) at the point \( P = (2, 3, \frac{3}{\sqrt{e}}) \).

**Solution:** Set \( F(x, y, z) = x^2 + z^2e^{y-x} - 13 \). Then the surface is \( F(x, y, z) = 0 \) and
\[
DF = (2x - z^2e^{y-x}, z^2e^{y-x}, 2ze^{y-x}),
\]
and
\[
DF(2, 3, \frac{3}{\sqrt{e}}) = (-5, 9, 6\sqrt{e}).
\]
An equation for the tangent plane at \( P \) is given by
\[
(x - 2, y - 3, z - \frac{3}{\sqrt{e}}) \cdot DF(2, 3, \frac{3}{\sqrt{e}}) = 0,
\]
or
\[
-5x + 3y + \frac{3}{\sqrt{e}}z = 19 + 9e^{-1}.
\]

2. Calculate the directional derivative in the direction \( v \) at the given point \( P \) for \( f(x, y, z) = x \ln(y + z) \), \( v = (2, -1, 1) \), and \( P = (2, e, e) \).

**Solution:** First we need to normalize \( v \) to a unit vector \( \vec{v} = \frac{1}{\sqrt{6}}(2, -1, 1) \). Then we have
\[
D_{\vec{v}}f(P) = Df(p) \cdot \vec{v} = \left. \left( \ln(y + z), \frac{x}{y + z}, \frac{x}{y + z} \right) \right|_{(2,e,e)} \cdot \frac{1}{\sqrt{6}} (2, -1, 1) = \frac{2}{\sqrt{6}}(1 + \ln 2).
\]

3. Use the chain rule to calculate the partial derivatives: \( \frac{\partial h}{\partial q} \) at \( (q, r) = (3, 2) \), where \( h(u, v) = ue^v \), \( u = q^3 \), \( v = qr^2 \).

**Solution:**
\[
\frac{\partial h}{\partial q} = \frac{\partial h}{\partial u} \frac{\partial u}{\partial q} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial q}
= e^v(3q^2) + ue^v(r^2)
= e^{qr^2}(3q^2 + q^3r^2)
\]
so that
\[
\frac{\partial h}{\partial q}(3, 2) = 135e^{12}.
\]
4. Use implicit differentiation to calculate the partial derivative: \( \frac{\partial w}{\partial z} \), where \( x^2w + w^3 + wz^2 + 3yz = 0 \).

**Solution:** \( w_z = -\frac{3y+2yw}{x^2+2^2+3w^2} \).

5. Find the critical points of the function, Then use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points: \( f(x, y) = x^3 + y^4 - 6x - 2y^2 \); \( g(x, y) = \ln x + 2\ln y - x - 4y \).

**Solution:**

a) \( f_x = 3x^2 - 6, \ f_y = 4y(y^2 - 1) \).

So the critical points are \((\pm \sqrt{2}, 0), (\pm \sqrt{2}, \pm 1)\).

\[ f_{xx} = 6x, \ f_{xy} = 0, \ f_{yy} = 12y^2 - 4. \]

At \((\sqrt{2}, 0)\), \( D = -24\sqrt{2} < 0, (\sqrt{2}, 0) \) is saddle point.
At \((-\sqrt{2}, 0)\), \( D = 24\sqrt{2} > 0. \) Since \( f_{xx}(-\sqrt{2}, 0) = -6\sqrt{2} < 0, (-\sqrt{2}, 0) \) is a local maxima point.
At \((\sqrt{2}, \pm 1)\), \( D = 48\sqrt{2} > 0. \) Since \( f_{xx}(\sqrt{2}, \pm 1) = 6\sqrt{2} > 0, (\sqrt{2}, \pm 1) \) is a local minima point.
At \((-\sqrt{2}, \pm 1)\), \( D = -48\sqrt{2} < 0, \) so \((-\sqrt{2}, \pm 1)\) is a saddle point.

b) \( g_x = \frac{1}{x} - 1; \ g_y = \frac{2}{y} - 4 \).

So the critical point is \( P = (1, \frac{1}{2}) \).

\[ g_{xx} = -\frac{1}{x^2}, \ g_{xy} = 0, \ g_{yy} = -\frac{2}{y^2}. \]

So that \( g_{xx}(1; \frac{1}{2}) = -1, g_{xy}(1; \frac{1}{2}) = 0, g_{yy}(1; \frac{1}{2}) = -8, D(1; \frac{1}{2}) = 8 > 0. \)

Thus \( (1, \frac{1}{2}) \) is a local maxima point.

6. Determine the global extreme values of the function on the given domain: \( f(x, y) = (4y^2 - x^2)e^{-x^2-y^2}, \ x^2 + y^2 \leq 2. \)

**Solution:**

a) Interior critical points:

\[ f_x = -2x(1 + 4y^2 - x^2)e^{-(x^2+y^2)}, \ f_y = -2y(-4 + 4y^2 - x^2)e^{-(x^2+y^2)}. \]

\( f_x(x, y) = f_y(x, y) = 0 \) iff \( (x, y) = (0, 0), (0, \pm 1), (\pm 1, 0) \). Hence the interior critical values are \( f(0, 0) = 0, f(0, \pm 1) = 4e^{-1} \), and \( f(\pm 1, 0) = -e^{-1} \).

b) Boundary extreme values: Since the boundary is \( x^2 + y^2 = 2 \), we have \( f(x, y) = e^{-2(5y^2 - 2)} \). It is easy to see that the maximum value is when \( y^2 = 2 \) so that \( f(x, y) = 8e^{-2} \), and the minimum value is when \( y = 0 \) so that \( f(x, y) = -2e^{-2} \).

c) The global maximum value is \( f(0, \pm 1) = 4e^{-1} \), and the global minimum value is \( f(\pm 1, 0) = -e^{-1} \).
7. Calculate the double integral
\[\int \int_R (xy^2 + \frac{y}{x}) \, dA\]
where
\[R = \{(x, y)|2 \leq x \leq 3, -1 \leq y \leq 0\}.

**Solution:**
\[= \int_{-1}^{3} \int_{-1}^{2} (xy^2 + \frac{y}{x}) \, dx \, dy\]
\[= \int_{-1}^{0} \left( \frac{x^2y^2}{2} + y \ln x \right)_{2}^{3} \, dy\]
\[= \int_{-1}^{0} \frac{5y^2}{2} + y \ln(\frac{3}{2}) \, dy\]
\[= \frac{5y^3}{6} + y \ln(\frac{3}{2}) \bigg|_{-1}^{0} = \frac{5}{6} - \frac{1}{2} \ln(\frac{3}{2}).\]

8. Use the polar coordinate to calculate the double integral
\[\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{1}{3 + x^2 + y^2} \, dx \, dy.\]

**Solution:** The domain is
\[0 \leq y \leq 1, \ y \leq x \leq \sqrt{1-y^2}.

In the polar coordinates, it can be written by
\[0 \leq r \leq 1, \ 0 \leq \theta \leq \frac{\pi}{4}.

The integral equals to
\[= \int_{0}^{\frac{\pi}{4}} \int_{0}^{1} \frac{1}{3 + r^2} r \, dr \, d\theta\]
\[= \frac{\pi}{8} \ln(3 + r^2) \bigg|_{r=0}^{1} = \frac{\pi}{8} \ln(\frac{4}{3}).\]

9. Evaluate the double integral
\[\int_{0}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2 y^2 \, dx \, dy.\]

**Solution:**
\[= \int_{0}^{2} \int_{0}^{\pi} r^4 \cos^2 \theta \sin^2 \theta \, r \, dr \, d\theta\]
\[= \left( \frac{1}{24} r^6 \right|_{0}^{2} \int_{0}^{\pi} \sin^2(2\theta) \, d\theta\]
\[= \frac{8}{3} \int_{0}^{\pi} 1 - \cos(4\theta) \, d\theta = \frac{4\pi}{3}.\]
10. Calculate the volume of the region above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = 1 \).

**Solution:** The region is bounded by

\[
x^2 + y^2 \leq \frac{1}{2}, \quad \sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - (x^2 + y^2)}.
\]

Hence the volume is given by

\[
= \int_{x^2+y^2\leq\frac{1}{2}} \left( \sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2} \right) dA
= \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} \left( \sqrt{1 - r^2} - r \right) r \, dr \, d\theta
= \frac{4 - \sqrt{2}}{3} \pi
\]

11. Find the mass of the region \( D \) that is enclosed by the cardioid \( r = 1 + \cos \theta \) with density \( \rho(x, y) = \sqrt{x^2 + y^2} \).

**Solution:**

\[
= 2 \int_0^{\pi} \int_0^{1+\cos \theta} \int_0^r r \, dr \, d\theta = \frac{2}{3} \int_0^{\pi} (1 + \cos \theta)^3 \, d\theta
= \frac{2}{3} \int_0^{\pi} (1 + 3 \cos \theta + 3 \cos^2 \theta + \cos^3 \theta) \, d\theta
= \frac{2\pi}{3} + \pi = \frac{5\pi}{3}.
\]

12. Use the Fubini’s theorem (or equivalently, the iterated integration) to evaluate the triple integral

\[
\int \int \int_E yz \cos(x^5) \, dV,
\]

where

\[
E = \{ (x, y, z) \mid 0 \leq 1 \leq 1, \ 0 \leq y \leq x, \ 0 \leq z \leq 2x \}.
\]

**Solution:** The integral equals

\[
= \int_0^1 \cos(x^5) (\int_0^x y \, dy) (\int_0^{2x} z \, dz) \, dx
= \int_0^1 x^4 \cos(x^5) \, dx = \frac{1}{5}.
\]

13. Use the spherical coordinates to calculate

\[
\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy.
\]
\[ \int_0^\pi \int_0^\pi \int_0^{2\pi} \rho^3 \sin^2 \phi \sin^2 \theta \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left( \frac{\rho^6}{6} \right) \left( \int_0^\pi \sin^2 \theta \, d\theta \right) \left( \int_0^{2\pi} \sin^3 \phi \, d\phi \right) = \frac{32\pi}{3} \left[ \frac{\cos^3 \phi}{3} - \cos \phi \right] \Bigg|_0^\pi = \frac{64}{9} \pi. \]

14. Find the center of mass for the lamina that occupies the region \( D \) and has the given density function \( \rho \): \( D \) is the triangular region with vertices \((0,0), (2,1), (0,3)\); \( \rho(x,y) = x + y \).

**Solution:** The domain is \( 0 \leq x \leq 1, \frac{1}{2}x \leq y \leq 3 - x \). First we calculate the mass

\[
m = \int_0^1 \int_{\frac{1}{2}x}^{3-x} (x + y) \, dy \, dx
= \int_0^1 \left( \frac{9}{2} - \frac{9}{8} x^2 \right) \, dx = \frac{33}{8}.
\]

Next we calculate the center of mass

\[
\int_0^1 \int_{\frac{1}{2}x}^{3-x} x(x + y) \, dy \, dx = \int_0^1 \left( \frac{9}{2} x - \frac{9}{8} x^3 \right) \, dx = \frac{63}{32},
\]

and

\[
\int_0^1 \int_{\frac{1}{2}x}^{3-x} y(x + y) \, dy \, dx = \int_0^1 \frac{9}{2} x - 3x^2 + \frac{x^3}{3} - \frac{(x - 3)^3}{3} \, dx = \frac{3}{4}.
\]

Hence

\[
\bar{x} = \frac{63}{132}, \quad \bar{y} = \frac{54}{33}.
\]

15. Evaluate the double integral by making an appropriate change of variables

\[
\int \int_R \frac{x + 2y}{\cos(x - y)} \, dxdy,
\]

where \( R \) is the parallelogram bounded by the lines \( y = x, \ y = x - 14, \ x + 2y = 0, \ x + 2y = 2 \).

**Solution:** Set

\[
u = x + 2y, \ v = x - y.
\]

Then we have

\[
0 \leq u \leq 2, \ 0 \leq v \leq 14.
\]

Solving \( x, y \) in terms of \( u, v \), we have

\[
x = \frac{u + 2v}{3}, \ y = \frac{u - v}{3}
\]

so that

\[
\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}.
\]
By the formula of change of variables, we have the integral equals to

\[
\frac{1}{3} \int_0^2 \int_0^{14} \frac{u}{\cos v} \, dudv = \frac{1}{3}(\int_0^2 u \, du)(\int_0^{14} \sec v \, dv)
\]

\[
= \frac{4}{3} \ln|\sec v + \tan v| \bigg|_0^{14} = \frac{4}{3} \ln(\sec(14) + \tan(14)).
\]

16. Use the map

\[
G(u, v) = (\frac{u + v}{2}, \frac{u - v}{2})
\]

to compute

\[
\int \int_{\mathcal{R}} ((x - y) \sin(x + y))^2 \, dxdy,
\]

where \(\mathcal{R}\) is the square with vertices \((\pi, 0), (2\pi, \pi), (\pi, 2\pi), \) and \((0, \pi)\).

**Solution:** Set \(x = \frac{u + v}{2}\) and \(y = \frac{u - v}{2}\). Then we have

\[u = x + y, \quad v = x - y.\]

The square is given by

\[\pi \leq x + y \leq 3\pi; \quad -\pi \leq x - y \leq \pi.\]

It is easy to see

\[
\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}
\]

so that the integral equals to

\[
\frac{1}{2} \int_{\{\pi \leq u \leq 3\pi, \quad -\pi \leq v \leq \pi\}} v^2 \sin^2 u \, dudv = \frac{1}{2}(\int_{-\pi}^{\pi} v^2 \, dv)(\int_{\pi}^{3\pi} \sin^2 u \, du) = \frac{\pi^4}{3}.
\]