Ceva’s Theorem

The three lines containing the vertices $A$, $B$, and $C$ of $\triangle ABC$ and intersecting opposite sides at points $L$, $M$, and $N$, respectively, are concurrent if and only if

\[
\frac{AN}{NL} \cdot \frac{BL}{LM} \cdot \frac{CM}{MC} = 1
\]

\[
\frac{NB}{BL} \cdot \frac{LC}{MC} \cdot \frac{MA}{NA} = 1
\]

Ceva’s Theorem

\[
\frac{K(\triangle ABL)}{K(\triangle ACL)} = \frac{BL}{LC}
\]

\[
\frac{K(\triangle PBL)}{K(\triangle PCL)} = \frac{BL}{LC}
\]
Ceva's Theorem

\[ \frac{BL}{LC} = \frac{K(\triangle ABL) - K(\triangle PBL)}{K(\triangle ACL) - K(\triangle PCL)} = \frac{K(\triangle ABP)}{K(\triangle ACP)} \]

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Ceva's Theorem

\[ \frac{CM}{MA} = \frac{K(\triangle BMC) - K(\triangle PMC)}{K(\triangle BMA) - K(\triangle PMA)} = \frac{K(\triangle BCP)}{K(\triangle BAP)} \]

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Ceva's Theorem

\[ \frac{AN}{NB} = \frac{K(\triangle ACN) - K(\triangle APN)}{K(\triangle BCN) - K(\triangle BPN)} = \frac{K(\triangle ACP)}{K(\triangle BCP)} \]
Ceva's Theorem

\[ \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = \frac{K(\triangle ACP)}{K(\triangle ABP)} \cdot \frac{K(\triangle BCP)}{K(\triangle BCP)} \cdot \frac{K(\triangle ACP)}{K(\triangle ACP)} = 1 \]

Now assume that

Let BM and AL intersect at P and construct CP intersecting AB at N, N' different from N.

Then AL, BM, and CN' are concurrent and

\[ \frac{AN'}{NB'} \cdot \frac{BL'}{LC'} \cdot \frac{CM'}{MA} = 1 \]

From our hypothesis it follows that

\[ \frac{AN'}{AN} = \frac{NB'}{NB} \]

So N and N' must coincide.
Medians

In \( \triangle ABC \), let \( M, N, \) and \( P \) be midpoints of \( AB, BC, AC \).

Medians: \( CM, AN, BP \)

Theorem: In any triangle the three medians meet in a single point, called the centroid.

\( M \) - midpoint \( \Rightarrow AM = BM \), \( N \) - midpoint \( \Rightarrow BN = CN \)
\( P \) - midpoint \( \Rightarrow AP = CP \)

\[ \frac{AM}{MB} \cdot \frac{BN}{NC} \cdot \frac{CP}{PA} = 1 \]

By Ceva’s Theorem they are concurrent.

Orthocenter

Let \( \triangle ABC \) be a triangle and let \( P, Q, \) and \( R \) be the feet of \( A, B, \) and \( C \) on the opposite sides.

\( AP, BQ, \) and \( CR \) are the altitudes of \( \triangle ABC \).

Theorem: The altitudes of a triangle \( \triangle ABC \) meet in a single point, called the orthocenter, \( H \).
Orthocenter

By AA
\(\triangle BRC \sim \triangle BPA\) (a right angle and \(\angle B\))
\[\Rightarrow \frac{BR}{BP} = \frac{BC}{BA}\]
\(\triangle AQB \sim \triangle ARC\) (a right angle and \(\angle A\))
\[\Rightarrow \frac{AQ}{AR} = \frac{AB}{AC}\]
\(\triangle CPA \sim \triangle CQB\) (a right angle and \(\angle C\))
\[\Rightarrow \frac{CP}{CQ} = \frac{AC}{BC}\]

\[
\begin{align*}
\frac{BR}{BP} \cdot \frac{AQ}{AR} \cdot \frac{CP}{CQ} &= \frac{BC}{BA} \cdot \frac{AB}{AC} \cdot \frac{AC}{BC} \\
&= 1
\end{align*}
\]

Orthocenter

By Ceva's Theorem, the altitudes meet at a single point.

Orthocenter

Traditional route:
\(BQ\) intersects \(AP\).
Now construct \(CH\) and let it intersect \(AB\) at \(R\).
Prove \(\triangle ARC \sim \triangle AQB\)
making \(\angle R = 90\).
Let \( \triangle ABC \) be a triangle and let \( AP, BQ, \) and \( CR \) be the angle bisectors of \( \angle A, \angle B, \) and \( \angle C. \)

**Angle Bisector Theorem:** If \( AD \) is the angle bisector of \( \angle A \) with \( D \) on \( BC \), then

\[
\frac{AB}{AC} = \frac{BD}{CD}
\]

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**Incenter**

Proof: Want to use similarity. Where is similarity?

Construct line through \( C \) parallel to \( AB \)

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**Incenter**

Proof: Want to use similarity. Where is similarity?

Construct line through \( C \) parallel to \( AB \)

Extend \( AD \) to meet parallel line through \( C \) at point \( E \).
Incenter

\[ \angle BAE \equiv \angle CEA \text{ - Alt Int Angles} \]
\[ \angle BDA \equiv \angle CDE \text{ - vertical angles} \]
\[ \triangle BAD \sim \triangle CDE \text{ - AA} \]

Therefore \[ \frac{AB}{CE} = \frac{BD}{CD} \]

Note that \[ \angle CEA \equiv \angle BAE \equiv \angle CAE \]
\[ \Rightarrow \triangle ACE \text{ isosceles} \Rightarrow CE = AC \text{ and} \frac{AB}{AC} = \frac{BD}{CD} \]

Incenter

Let \( \triangle ABC \) be a triangle and let AP, BQ, and CR be the angle bisectors of \( \angle A, \angle B, \) and \( \angle C. \)

Theorem: The angle bisectors of a triangle \( \triangle ABC \) meet in a single point, called the incenter, I.

Proof: Angle bisector means:
\[ \frac{AB}{AC} = \frac{BP}{PC} \]
\[ \frac{BA}{BC} = \frac{AQ}{QC} \]
\[ \frac{CA}{CB} = \frac{AR}{RB} \]

By Ceva's Theorem we need to find the product:
\[ \frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} \]
Incenter

\[
\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = \frac{AC}{BC} \cdot \frac{AB}{AC} \cdot \frac{BC}{AB} = 1
\]

Thus by Ceva's Theorem the angle bisectors are concurrent.

Circumcenter & Perpendicular Bisectors

Does Ceva's Theorem apply to perpendicular bisectors?

Circumcenter & Perpendicular Bisectors

How can we get Ceva's Theorem to apply to perpendicular bisectors?
Circumcenter & Perpendicular Bisectors

Draw in midsegments

EF || BC \Rightarrow perpendicular bisector of BC is perpendicular to EF \Rightarrow is an altitude of \Delta DEF

Perpendicular bisectors of AB, BC and AC are altitudes of \Delta DEF.

Altitudes meet in a single point \Rightarrow perpendicular bisectors are concurrent.

Circumcircle

Theorem: There is exactly one circle through any three non-collinear points.

The circle = the circumcircle
The center = the circumcenter, O.
The radius = the circumradius, R.

Theorem: The circumcenter is the point of intersection of the three perpendicular bisectors.
Question

Where do the perpendicular bisectors of the sides intersect the circumcircle?

At one end is point of intersection of angle bisector with circumcircle
The other end is point of intersection of exterior angle bisector with circumcircle.

Extended Law of Sines

Theorem: Given \( \triangle ABC \) with circumradius \( R \), let \( a, b, \) and \( c \) denote the lengths of the sides opposite angles \( \angle A, \angle B, \) and \( \angle C \), respectively. Then

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R
\]
Three cases:

Case I: \( \angle A < 90^\circ \)
BP = diameter
\( \Rightarrow \Delta BCP \) right triangle
BP = 2R
\( \Rightarrow \sin P = a/2R \)
\( \angle A = \angle P \)
\( \Rightarrow 2R = a/\sin A \)

Case II: \( \angle A > 90^\circ \)
BP = diameter
\( \Rightarrow \Delta BCP \) right triangle
BP = 2R
\( \Rightarrow \sin P = a/2R \)
\( \angle A = \angle P \)
\( \Rightarrow 2R = a/\sin A \)
Case III: $\angle A = 90^\circ$
BP = a = diameter
BP = 2R
2R = a = a/sin A

Proof

Circumradius and Area

Theorem: Let R be the circumradius and K be the area of \( \triangle ABC \) and let a, b, and c denote the lengths of the sides as usual. Then \( 4KR = abc \)

\[ K = \frac{1}{2} ab \sin C \]
\[ 2K = ab \sin C \]
\[ c/\sin C = 2R \]
\[ \sin C = c/2R \]
\[ 2K = abc/2R \]
\[ 4KR = abc \]