Last time:

Exam 1: Tues Oct 18 @ 7 PM
Covers 14.6 - 14.8, Ch. 13, 16.1 - 16.3

Nathan (DD1 & DD6) 116 Roger Adams Lab
Yi (DD2 & DD3) 1404 Siebel
Michael (DD4, DD5, & DD7) 1320 DCL
& Caglar

Monday OH 11:00 - 12:30
(not Friday)

First Recognition Theorem: If $F$ is a continuous vector field on an open, connected region $D$ in $\mathbb{R}^2$. Then

$$ \int_{C'} F \cdot dr = \begin{cases} \text{path-independent in } D \quad \text{if } F \text{ is conservative on } D. \\ \end{cases} $$

Works exactly the same for $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ or $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

Second Recognition Theorem: If $F = P \vec{e}_x + Q \vec{e}_y$, $P$ & $Q$ have continuous partials on an open simply connected domain.

$$ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{if } F \text{ is conservative.} $$
Generalization to $F: \mathbb{R}^3 \to \mathbb{R}^3$? $F = P_1 + Q_2 + R_3$

Need $P_y = Q_x$ and $P_z = R_x$ and $Q_z = R_y$

(say "curl" $F = \mathbf{0}$. Will return to this in §16.5)

Next goal "Green's theorem". First need "double integrals".

§ 15.1 Double Int's.

Goal: Given function $F: \mathbb{R}^2 \to \mathbb{R}$ use integration to measure volume of region under graph.

For simplicity, consider $F$ on rectangle $[a,b] \times [c,d]$

First, review $\int_a^b f(x)\,dx$.

Divide $[a,b]$ into small subintervals $[x_{i-1}, x_i]$ (equal length).

In $[x_{i-1}, x_i]$ take sample point $x_i^*$.

Approximate area by $f(x_i^*) (x_i - x_{i-1}) = \frac{f(x_1^*) (x_1 - x_0)}{(b-a)/n} + \frac{f(x_2^*) (x_2 - x_1)}{(b-a)/n} + \cdots + \frac{f(x_n^*) (x_n - x_{n-1})}{(b-a)/n}$.

$\int_a^b f(x)\,dx \approx \lim_{n \to \infty} \text{above sum.}$. ("Riemann sum").

Same technique for $F: \mathbb{R}^2 \to \mathbb{R}$.

Subdivide rectangle $R = [a,b] \times [c,d]$ by subdividing $[a,b]$ and $[c,d]$.

Take one sample point $(x_{25}^*, y_{25}^*)$ in $R_{25} = [x_1, x_2] \times [y_1, y_5]$. 


Consider double summation
\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]
\[
\frac{b-a}{m} \cdot \frac{d-c}{n} = \Delta x \Delta y
\]

Define \underline{double integral}
\[
\iint_R f \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A
\]

This measures the \underline{volume} of the solid under graph \( f \), over \( R \).

But how do we compute it?

\[ \text{§ 15.2 \ Reducing to single-variable integrals by considering "slices".} \]

\[
\text{Ex : } R = [0,2] \times [0,1] \quad , \quad f(x,y) = x^2 + y^2
\]

Find \( \iint_R f(x,y) \, dA \).

"Add up" area of "slices" where \( y \) is fixed.
\[
A(y) = \text{area of } y \text{-slice } = \int_0^2 x^2 + y^2 \, dx = x^3 + xy^2 \bigg|_0^2 = \frac{8}{3} + 2y^2
\]

Now add up all areas →
\[
\text{Vol} = \int_0^1 A(y) \, dy = \int_0^1 \left( \frac{8}{3} + 2y^2 \right) \, dy = \frac{8y}{3} + \frac{2y^3}{3} \bigg|_0^1 = \frac{8}{3} + 2/3 = 10/3
\]

\( \therefore \) we could first slice in \( x \) direction, then

\[
\text{take } \int_0^2 A(x) \, dx
\]

\[
A(x) = \int_0^1 x^2 + y^2 \, dy = x^2y + \frac{y^3}{3} \bigg|_0^1 = x^2 + \frac{1}{3}
\]

Vol = \( \int_0^2 A(x) \, dx = \int_0^2 x^2 + \frac{1}{3} \, dx = \frac{x^3}{3} + \frac{x}{3} \bigg|_0^2 = \frac{8}{3} + \frac{2}{3} = \frac{10}{3} \).
**Theorem (Fubini)** Assume $f$ continuous on $R = [a,b] 	imes [c,d]$

Then $\iint_R f(x,y) \, dx \, dy = \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy = \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx$.

Known as "iterated integrals".

**Note** Sometimes, one formulation more convenient than other.

**Ex** $f(x,y) = \frac{x}{1+xy}$  $R = [0,1] \times [0,1]$  

Try $\int_0^1 \int_0^1 \frac{x}{1+xy} \, dx \, dy$ means integrate first w.r.t. $x$.

Tricky!

Try instead $\int_0^1 \int_0^1 \frac{x}{1+xy} \, dy \, dx$

Let $u = 1 + xy$ (constant)

$du = x \, dy$

$= \int_0^1 \int_0^1 \frac{1+xy}{u} \, dx \, dy = \int_0^1 \ln(1+xy) - \ln(1+xy) \, dx$

Integrate by parts $u = \ln(1+xy)$  $v = 1+x$

$du = \frac{dx}{1+xy}$  $dv = dx$

$= uv - \int v \, du$

$= \ln(1+x) \bigg|_0^1 - \int_0^1 \frac{1+xy}{1+xy} \, dx$

$= (\ln 2) 2 - (\ln 1) 1 - 1 - 0$

$= 2 \ln 2 - 1 - 1 = \ln 4 - 1$

Next time $\iint_D f(x,y) \, dA$

**General region.**

Uses: 1) $\iint_D 1 \, dA = \text{area } D$

2) average value of $f$ on $D = \frac{1}{\text{area } D} \iint_D f(x,y) \, dA$. 