Last time Green's theorem \( F = (P, Q) \)
\[
\oint_{\partial D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dx = \iint_{D} \nabla \cdot F \, dA
\]

Used to replace line int by easier double int
or double int by easier line int

Recall If \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \) on simply conn domain \( \Omega \) then
\( F \) is conservative.

This follows from Green's Thm:

Enough to show \( \iint_{\partial C} F \cdot d\vec{r} = 0 \) for every loop \( C \) in \( \Omega \).

\( \Omega \) simply conn \( \Rightarrow \) \( C \) is boundary of some region \( \partial D \subset \Omega \)

Then \( \iint_{\partial C} F \cdot d\vec{r} = \iint_{D} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = 0 \)

Next few lectures: Surfaces & Surface Integrals
5.16.6 Parametric surfaces.

Parametric curves: \( \vec{r}: [a, b] \to \mathbb{R}^3 \)
\( C = \vec{r}([a, b]) \)

Parametric surfaces: \( \vec{r}: D \to \mathbb{R}^3 \)
\( \text{some region } \Delta \subset \mathbb{R}^2 \)

Example Cylinder: \( \vec{r}(s, t) = (s, \cos t, \sin t) \)
Ex. Northern hemisphere

\[ \vec{F}(u,v) = (v \cos u, v \sin u, \sqrt{1 - v^2}) \]

\[ \vec{F}(u,v) = (u, v, \sqrt{1-u^2-v^2}) \] gives another parametrization of northern hemisphere.

One more parametrization of entire unit sphere comes from spherical coordinates:

\[ \vec{F}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \]
How do find tangent planes

Recall tangent plane to level surface \( f(x, y, z) = 0 \) at \((a, b, c)\) has normal vector \( \nabla f(a, b, c) \), so plane described by \( \nabla f(a, b, c) \cdot (x-a, y-b, z-c) = 0 \).

Example Sphere given by \( x^2 + y^2 + z^2 = 1 \)

\( \nabla f = (2x, 2y, 2z) = \) scalar mult \( f \) \((x, y, z)\)

Plane given by \( a(x-a) + b(y-b) + c(z-c) = 0 \)

or \( ax + by + cz = a^2 + b^2 + c^2 = 1 \).

What about tangent plane to parametrized surface?

\( f(\vec{r}) = \) constant, so

\[ \frac{\partial}{\partial u} (f(\vec{r})) = \frac{\partial}{\partial u} (f(\vec{r})) = 0. \]

Chain rule:

\[ \frac{\partial}{\partial u} (f(\vec{r})) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} \]

\[ = \nabla f \cdot \frac{\partial \vec{r}}{\partial u} \]

\[ \frac{\partial \vec{r}}{\partial u} \cdot \nabla f = 0, \text{ so } \frac{\partial \vec{r}}{\partial u} \text{ lies on tangent plane.} \]

Same for \( \frac{\partial \vec{r}}{\partial v} \).

The vector \( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \) gives normal vector for tangent plane.

Example Back to sphere.

Using \( \vec{r}(u, v) = (vcosu, vsinu, \sqrt{1-v^2}) \),
\[
\frac{\partial \vec{r}}{\partial u} = (-\sin u, \cos u, 0)
\]
\[
\frac{\partial \vec{r}}{\partial v} = (\cos u, \sin u, -v) / \sqrt{1-v^2}
\]

\[
\text{normal vector: } \begin{vmatrix} \hat{t} & \hat{s} & \hat{n} \\ -\sin u & \cos u & 0 \\ \cos u & \sin u & -v / \sqrt{1-v^2} \end{vmatrix} = \frac{-v^2 \cos u}{\sqrt{1-v^2}} \hat{t} - \frac{v^2 \sin u}{\sqrt{1-v^2}} \hat{s} + (-v \sin^2 u - v \cos^2 u) \hat{n}
\]

\[
= \left( -\frac{v^2 \cos u}{\sqrt{1-v^2}}, -\frac{v^2 \sin u}{\sqrt{1-v^2}}, -v \right)
\]

\[
\left( \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \right)
\]

Using \( \vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \)

Find \( \frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta} = \sin \phi \hat{n}(\phi, \theta) \).

Find tangent plane to surface param by \( \vec{r}(u, v) = (u^2, u \sin v, u \cos v) \) at \( \vec{r}(1, 0) = (1, 0, 1) \)

\[
\frac{\partial \vec{r}}{\partial u} = (2u, \sin v, \cos v)
\]
\[
\frac{\partial \vec{r}}{\partial v} = (0, u \cos v, -u \sin v)
\]

\[
\hat{n} = \begin{vmatrix} \hat{t} & \hat{s} & \hat{n} \\ 2u & \sin v & \cos v \\ 0 & u \cos v & -u \sin v \end{vmatrix} = (-u(\sin v + u \cos v), 2u^2 \sin v, 2u^2 \cos v)
\]

\( \hat{n} = u(1, -1, -1) \quad \text{at } u=1, v=0 \) get \( \hat{n} = (-1, 0, 2) \)

Tangent plane: \( \hat{n} \cdot (x-1, y, z-1) = 0 \) or \(-x + 2z = 1\)
Note: \( \frac{\partial \vec{r}}{\partial v} (0,v) = \vec{0} \) so can't see tangent plane at \( \vec{r} (0,v) = (0,0,0) \).

The surface is the elliptic paraboloid \( x = y^2 + z^2 \).

Under parameterization \( \vec{r} (u,v) = (u^2 + v^2, u, v) \)

\[ \frac{\partial \vec{r}}{\partial u} = (2u, 1, 0), \quad \frac{\partial \vec{r}}{\partial v} = (2v, 0, 1) \]

\[ \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (1, -2u, -2v) \]

So tangent plane at \( (a,b,c) \) given by

\[ (1, -2b, -2c) \cdot ((x-a, y-b, z-c) = 0 \]

or \( x - 2by - 2cz = a - 2b^2 - 2c^2 \).

Tangent plane @ \( (0,0,0) \) given by \( x = 0 \).

@ \( (1,0,0) \) given by \( x - 2z = -1 \).

Next time: surface area, surface integrals.