Math 351 - Elementary Topology
Wednesday, October 24  ** Products and the Hausdorff condition

1. Show that if $X$ and $Y$ are Hausdorff spaces, then so is their product $X \times Y$.

2. Show that $X$ is Hausdorff if and only if the diagonal subset
   \[ \Delta(X) = \{(x, y) \in X^2 \mid x = y\} \subseteq X \times X \]
   is closed.

3. Let $f, g : X \to Y$ be continuous, and suppose that $Y$ is Hausdorff. Show that if $D \subseteq X$ is dense in $X$ and $f(d) = g(d)$ for all $d \in D$, then necessarily $f(x) = g(x)$ for all $x \in X$.
   Hint: This was on a previous worksheet, but it now follows easily from problem 2 above.

Write your answer(s) on the rest of this sheet (and back).

Solutions.

1. Let $(x, y)$ and $(x', y')$ be distinct points in $X \times Y$. This means that either $x \neq x'$ or $y \neq y'$ (or both). Without loss of generality, we assume $x \neq x'$. Then, since $X$ is Hausdorff, this means we can find disjoint neighborhoods $U$ and $U'$ of $y$ and $x'$ in $X$. It follows that $U \times Y$ and $U' \times Y$ are disjoint neighborhoods of $(x, y)$ and $(x', y')$.

2. $(\Rightarrow)$ Assume $X$ is Hausdorff. We will show that $X \times X \setminus \Delta(X)$ is open. Let $(x, y) \notin \Delta(X)$. This means that $x \neq y$. So there must be disjoint neighborhoods $U$ and $V$ of $y$. Then $U \times V$ is a neighborhood of $(x, y)$ in $X \times X$, and it does not meet $\Delta(X)$, since if $(z, z) \in \Delta(X) \cap (U \times V)$, this would mean that $z \in U \cap V$, which would contradict that $U \cap V = \emptyset$.

   $(\Leftarrow)$ Assume $\Delta(X)$ is closed. Let $x \neq y$ be distinct points in $X$. Then $(x, y)$ lies in the open set $(X \times X) \setminus \Delta(X)$. By the definition of the product topology, this means that we can find a basic open set $U \times V$ with
   \[ (x, y) \in U \times V \subseteq (X \times X) \setminus \Delta(X). \]
   The fact that $U \times V$ misses the diagonal says precisely that $U \cap V$ is empty.

3. Let $f$ and $g$ be as described in the statement of the problem. Consider $\Delta(Y) \subseteq Y \times Y$. Since $Y$ is Hausdorff, by problem 2 we know this diagonal subset is closed. The function $(f, g) : X \to Y \times Y$ is continuous by the universal property of products. We conclude that
   \[ (f, g)^{-1}(\Delta(Y)) = \{x \in X \mid f(x) = g(x)\} \]
   is closed in $X$. Since this closed set contains the dense set $D$, this closed set must be all of $X$. 