Problem 1. (i) Let \((X, d)\) be a metric space and let \(x \in X\). Define \(d_x : X \to \mathbb{R}\) by \(d_x(y) = d(x, y)\). Show that \(d_x\) is continuous \((X\) is equipped with the metric topology).  
(ii) Use the above to show that \(S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 = 1\} \subseteq \mathbb{R}^n\) is closed.

Problem 2. Let \(X = \mathbb{R} \cup \{z\}\). Define a proper subset to be closed if it is finite and does not contain the point \(z\). Show that this defines a topology on \(X\) and that the point \(z\) is dense in \(X\) (such a point is called a generic point).

Problem 3. Let \(A\) and \(B\) denote subsets of a topological space \(X\).  
(i) Show that \(A \cup B = \overline{A \cup B}\).  
(ii) Show that \(\bigcup A_i \subseteq \bigcup \overline{A_i}\) and give an example these two subsets are not equal.

Problem 4. Let \(f\) and \(g\) be continuous functions \(X \to \mathbb{R}\). Show that the subset \(W \subseteq X\) of points \(x\) such that \(f(x) = g(x)\) is closed. This shows that a real-valued continuous function is determined by its values on a dense subset.

Problem 5. Suppose \(f : X \to Y\) is continuous and let \(A \subseteq X\) be a subset. If \(x\) is a limit point of \(A\), is \(f(x)\) a limit point of \(f(A)\)?

Problem 6. Show that if \(W \subseteq X\) is any subset, then  
\[W^o = X \setminus (X \setminus W)\]

Problem 7. Show that \(f : X \to Y\) is continuous if and only if for every closed \(Z \subseteq Y\), \(f^{-1}(Z) \subseteq X\) is closed.
Problem 8. Let $C$ and $D$ be sets, and suppose given functions $f : C \to D$ and $g : D \to C$.

(i) Show that if $g \circ f$ is the identity function of $C$, then $f$ is injective.

(ii) Show that if $g \circ f$ is the identity function of $C$, then $g$ is surjective.

(iii) Use the above to conclude that $f$ is bijective if and only if it is a (categorical) isomorphism (i.e., there exists a function $g : D \to C$ such that $f \circ g$ is the identity on $D$ and $g \circ f$ is the identity on $C$).

Note that the above shows, by forgetting about the topologies, that a homeomorphism is a bijection.

Problem 9. Show that $\mathbb{R}$ is homeomorphic to $(0, 1)$. (Hint: you should be able to use functions of the form $\frac{ax+b}{cx+d}$ for appropriate choices of $a, b, c, \text{and } d$.)

Problem 10. (a) Let $X = \mathbb{R} \cup \{\infty\}$, topologized as follows: if a subset $W$ does not contain the point $\infty$, then it is open if it is open in the usual topology on $\mathbb{R}$; if $W$ does contain $\infty$, then $W$ is open if $X \setminus W$ is closed in $\mathbb{R}$ (under the usual topology) and contained in some closed interval $[a, b]$. Show this defines a topology on $X$.

(b) Show that the space $X$ from part (a) is homeomorphic to $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

Problem 11. Define a function $f : [0, 1) \cup \{2\} \to [0, 1]$ to be the inclusion $[0, 1) \hookrightarrow [0, 1]$ and $f(2) = 1$. Show that $f$ is a continuous bijection but not a homeomorphism.