Problem 1. (The Cantor set) Let $C_0 = I$. For $k \geq 1$, define $C_k \subseteq I$ inductively by

$$C_k = C_{k-1} \cap \left( \left[ 0, \frac{1}{3^k} \right] \cup \left[ \frac{2}{3^k}, \frac{3}{3^k} \right] \cup \left[ \frac{4}{3^k}, \frac{5}{3^k} \right] \cup \cdots \cup \left[ \frac{3^k-1}{3^k}, 1 \right] \right)$$

(so each $C_k$ is a union of intervals, and each $C_k$ is obtained from $C_{k-1}$ by removing the “middle thirds” of the intervals). Define the cantor set by $C = \bigcap_k C_k$.

Show that $C$ is totally disconnected. (Hint: what do the components of each $C_k$ look like?)

Problem 2. Give an example of a locally connected $X$ and a continuous $f : X \to Y$ such that $f(X)$ is not locally connected.

Problem 3. Let $X = \mathbb{N}$, equipped with the cofinite topology (see HW1, problem 7). Show that $X$ is connected and locally connected, but not path connected or locally path connected.

Problem 4. Show that if $X$ is Hausdorff, then limits of sequences in $X$ are unique. That is, if $\{x_n\}$ converges to both $x$ and $y$, then $x = y$.

Problem 5. (i) Show that a space $X$ is Hausdorff if and only if the diagonal $\Delta(X) \subseteq X \times X$ is closed. By HW3, problem 1(iii), this shows that if $Y$ is any space and $f, g$ are continuous functions $Y \to X$ which agree on a dense subset of $Y$, then $f = g$.

(ii) Show that if $Y$ is Hausdorff and $f : X \to Y$ is continuous, then the graph of $f$ is closed in $X \times Y$.

Problem 6. A subset $A \subseteq X$ is said to be **locally closed** if it can be written as the intersection of an open set and a closed set.

(i) Show that for any space $X$, the diagonal $\Delta(X) \subseteq X \times X$ is always locally closed.
(Hint: Say a point \((x, y) \in X \times X\) is “bad” if \(x\) and \(y\) do not satisfy the Hausdorff property; that is, \((x, y)\) is bad if every pair of neighborhoods \(U_x\) and \(U_y\) intersect nontrivially. Show that if \(B\) denotes the set of bad points, then \(\Delta(X) = \Delta(X) \cup B\). Then show that \(B\) is closed by showing that it contains its accumulation points.)

(ii) Conclude that for a continuous map \(f : X \to Y\) between arbitrary spaces \(X\) and \(Y\), the graph of \(f\) is locally closed in \(X \times Y\).

**Problem 7.** (The line with doubled origin) Let \(Y\) be the quotient of \(\mathbb{R} \times \{-1, 1\}\) by the relation \((x, -1) \sim (x, 1)\) if \(x \neq 0\).

(i) Show that \(Y\) is not Hausdorff.

(ii) Define \(f, g : \mathbb{R} \to Y\) by \(f(x) = (x, 1)\) and \(g(x) = (x, -1)\). Show that \(f\) and \(g\) are both continuous. Note, however, that they agree on the dense subset \(\mathbb{R} \setminus \{0\}\) but are clearly not equal.