Part IV

Week 3A: Representation Theory of Groups
Objects can be understood by their actions on simpler objects

Representation of finite groups studies actions of the group on finite dimensional vector spaces

Representations of groups “linearizes” the group structure

Representations are occasionally required for deeper positive characteristic results
A **representation** of a finite group $G$ is a group homomorphism from $G$ to the automorphism group of a finite dimensional vector space $V$.

The vector space with basis $G$ has an obvious representation where $g$ takes the $h$th basis vector to $hg$. This is called the **regular representation**.

In fact, it has a natural ring structure called the **group ring**, $kG$.

A **module** is a ring homomorphism from $kG$ to the endomorphism ring of a finite dimensional vector space.

Modules and representations are the same thing.

Even simpler, we can just specify one invertible matrix per generator of the group.
We give the example of $S_3$ acting on a three dimensional space as a permutation of the standard basis. Define

\[ \pi : S_3 \to \text{GL}(3, k) \]

by:

\[
\begin{array}{c}
[1, 2, 3] \mapsto \begin{pmatrix}
1 & . & . \\
. & 1 & . \\
. & . & 1
\end{pmatrix} &
[2, 1, 3] \mapsto \begin{pmatrix}
1 & . & . \\
. & 1 & . \\
. & . & 1
\end{pmatrix} &
[1, 3, 2] \mapsto \begin{pmatrix}
1 & . & . \\
. & . & 1 \\
. & 1 & .
\end{pmatrix}

\end{array}
\]

\[
\begin{array}{c}
[3, 1, 2] \mapsto \begin{pmatrix}
. & . & 1 \\
1 & . & . \\
. & 1 & .
\end{pmatrix} &
[2, 3, 1] \mapsto \begin{pmatrix}
. & 1 & . \\
1 & . & . \\
. & . & 1
\end{pmatrix} &
[3, 2, 1] \mapsto \begin{pmatrix}
. & . & 1 \\
1 & . & . \\
1 & . & .
\end{pmatrix}
\end{array}
\]

The module structure takes

\[
\sum_{g \in S_3} \alpha_g e_g \in k^G \mapsto \sum_{g \in S_3} \alpha_g \pi(g) \in \text{End}(k^3)
\]
A submodule of $\pi : kG \to End(V)$ is a restriction
\[ \sigma : kG \to End(W) \] where $\sigma(w) = \pi(w) \in W \leq V$ for all $w \in W$.

Everyone abbreviates the name of $\pi$ to “V” and the name of $\sigma$ to “W” to make it easier to talk about elements.

If we form a basis of $V$ by extending a basis of $W$ then then the matrices have the form
\[ \pi(g) = \begin{pmatrix} \sigma(g) & . \\ B(g) & C(g) \end{pmatrix} \]

$B(g)$ is “trash” but $C(g)$ defines a representation $C : G \to Aut(V/W)$ called a quotient module.
Direct sums

- If $V = U \oplus W$ with both $U$, $W$ submodules, then write a basis for $V$ by concatenating bases for $U$ and $W$.

- In this basis we have the even simpler form

$$
\pi(g) = \begin{pmatrix}
\sigma(g) \\
\cdot \\
C(g)
\end{pmatrix}
$$

where $\sigma$ defines the submodule structure of $U$ and $C$ defines the quotient module structure of $V/U \cong W$

- This is nicer because $B(g) = 0$ can be ignored. $B(g)$ describes the interaction of $U$ and $W$ and makes life difficult if it is nonzero.

- We say that $V$ is **decomposable** or a **direct sum** in this case where $B(g) = 0$

- If $U \not\cong W$ then virtually every question about $V$ is simply the disjoint union of the same question for $U$ and $W$.
Atomic power

- Many areas of science try to break things down into pieces which cannot be broken down further.

- A module is **irreducible = simple** if it has no nonzero proper submodules.

- A module is **indecomposable** if it is not a direct sum of two nonzero proper submodules.

- To understand the module theory means to understand the indecomposable modules.

- This is possible for human minds iff the Sylow $p$-subgroup is cyclic, dihedral, or $p$ does not divide $|G|$.

- In the latter case, indecomposable = irreducible and the theory is trivial (direct product of fields).

- This is our case and why character theory usually suffices.
Example again

- Our representation of $S_3$ on $k^3$ is not irreducible, the subspace spanned by $\langle 1, 1, 1 \rangle$ forms a submodule.

- We extend this to a basis

  $$e_1 = \langle 1, 1, 1 \rangle \quad e_2 = \langle 0, 1, -1 \rangle \quad e_3 = \langle -1, 0, -1 \rangle$$

- Let $g = [2, 1, 3]$, then $e_1 \pi(g) = e_1$, $e_2 \pi(g) = \langle 1, 0, -1 \rangle = -2e_1 - 2e_2 - 3e_3$, and $e_3 \pi(g) = \langle 0, -1, -1 \rangle = -2e_1 + e_2 - 2e_3$, so in the new basis

  $$\pi(g) \mapsto \begin{pmatrix} 1 \\ -2 \\ -2 \\ -2 \\ 1 & -2 \end{pmatrix}$$

- Submodule is top left, quotient module is bottom right, but trashy bottom left is the cohomology.
Example yet again

- We know from theory the cohomology vanishes except for $p = 3$, but we want a basis that shows this.

- Let $f_1 = \langle 1, 1, 1 \rangle$, $f_2 = \langle 1, -1, 0 \rangle$, $f_3 = \langle 0, 1, -1 \rangle$ which is a basis if the characteristic is not 3, and we get the new matrices:

  $[1, 2, 3] \mapsto \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix}$
  
  $[2, 1, 3] \mapsto \begin{pmatrix} 1 & . & . \\ . & -1 & . \\ . & 1 & 1 \end{pmatrix}$
  
  $[1, 3, 2] \mapsto \begin{pmatrix} 1 & . & . \\ . & 1 & 1 \\ . & . & -1 \end{pmatrix}$
  
  $[3, 1, 2] \mapsto \begin{pmatrix} 1 & . & . \\ . & -1 & -1 \\ . & 1 & . \end{pmatrix}$
  
  $[2, 3, 1] \mapsto \begin{pmatrix} 1 & . & . \\ . & . & 1 \\ . & -1 & -1 \end{pmatrix}$
  
  $[3, 2, 1] \mapsto \begin{pmatrix} 1 & . & . \\ . & . & -1 \\ . & -1 & . \end{pmatrix}$

- Notice the block structure $\sigma(g) = [1]$ and $C(g)$ the bottom right $2 \times 2$ corner.
Roughly speaking we do exactly the same thing as we did with characters:

1. Find the permutation representations on the cosets of Young subgroups (row stabilizers of Young tableaux = centralizers of Young tabloids)

2. Use Gram-Schmidt to break down the representations into direct sums of previously known simple representations and one copy of the unique new simple representation

- Gram-Schmidt requires subtraction!
- Subtracting modules is hard.
- Polytabloids (=linear combinations of row-equivalence classes of Young tableaux) form a natural basis of that unique new simple module, called the **Specht module**
Conclusion

- Representations = modules linearize a group
- Irreducible = simple ones form building blocks
- Indecomposables form the real buildings (or pyramids at least)
- For us, indecomposable = irreducible
- Breaking down representations into irreducibles is very hard, so we need to use the polytabloids to form a basis

THE END