PST, just-non-PT groups

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Outline

- Intro to just-non-PT groups
- Examples of just-non-PT groups
- Classifications of just-non-PT groups
- Application and a surprising example
- Current work on PST, just-non-PT
Let $\mathcal{X}$ be a class of groups we want to understand.

Study groups that are \textbf{just} barely \textbf{not} in $\mathcal{X}$

\textbf{just-non}$\mathcal{X} = \text{not in } \mathcal{X}$, but every proper quotient is

Just-non-trivial groups = simple groups

1960: Newman: countable, soluble, just-non-\textbf{abelian}
1970: McCarthy and Wilson: just-non-\textbf{finite}
1973: Robinson: soluble, just-non-\textbf{T}
1982: Longobardi: finite, nilpotent, just-non-\textbf{PT}
2009: Working on finite, PST, just-non-\textbf{PT}
Examples of finite, non-cyclic, but every proper quotient is cyclic:

- Klein four, or any \textbf{elementary abelian group of rank 2}
- any dihedral group of order $2p$
- $A_4$ the alternating group on four points, or any $\text{AGL}(1, q)$
- $S_5$, or any symmetric group on at least five points
- the \textbf{automorphism group of} $M_{11} \times M_{11}$
Classification: finite, just-non-cyclic

- Three types:
  1. Elementary abelian $p$-group of rank 2
  2. $H(n, p) \leq AGL(1, p^k)$ for $n > 1$, $p$ prime, $k = \text{Order}(p \mod n)$
  3. $S^k \leq G \leq \text{Aut}(S^k) = \text{Aut}(S) \wr \text{Sym}(k)$ with $G/S^k$ cyclic and acting transitively on $\{1, \ldots, k\}$

- Second type takes an element of order $n$ in $GF(p^k)^\times$ acting on the additive group of $GF(p^k)$.

- Standard example of just-non-$\mathcal{X}$: Semidirect of $\mathcal{X}$-group acting faithfully and irreducibly on some other group
Several classes of groups are defined based on various ideas of “normality” being equal in a group:

- **T-group** = subnormal subgroups are normal
  \[ H \trianglelefteq \ldots \trianglelefteq G \iff H \trianglelefteq G, \text{ that is } gH = Hg \text{ for all } g \in G \]

- **PT-group** = subnormal subgroups are permutable
  \[ H \trianglelefteq \ldots \trianglelefteq G \iff \langle g \rangle H = H\langle g \rangle \text{ for all } g \in G \]

- **PST-group** = subnormal subgroups are Sylow permutable
  \[ H \trianglelefteq \ldots \trianglelefteq G \iff PH = HP \text{ for all Sylows } P \leq G \]
Examples of PT-groups

- \( T \implies PT \implies PST \)

- Every abelian group is \( T \), \( PT \), and \( PST \)

- Every nilpotent group is \( PST \)

- Nilpotent \( T \) groups are abelian or Hamiltonian \((Q_8 \times O \times E)\)

- Dihedral groups of order \( 2p \) are \( T \), but not abelian

- \( p \rtimes p^2 \) is \( PT \), but not \( T \)

- Dihedral group of order 8 is \( PST \), but not \( PT \)
Example of just-non-PT groups

- Extraspecial groups like $D_8$ are finite, nilpotent, just-non-abelian, just-non-T, just-non-PT, PST

- $D_8 \rtimes C_{2^n}$ is also finite, nilpotent, just-non-abelian, just-non-T, just-non-PT, PST

- $S_3 \times C_3$ is finite, supersoluble, just-non-T, just-non-PT, just-non-PST

- $(C_3 \times C_9) \rtimes 7^3$ is finite, soluble, not supersoluble, just-non-PT, just-non-PST
Classifications

- Finite, soluble, PST, just-non-PT groups $\equiv$ just-non-modular $p$-groups of Longobardi 1982
  (reduction shown to me by Matt Ragland)

- Finite, supersoluble, non-nilpotent, just-non-PT groups $\equiv$
  some just-non-T types of Robinson 1973 (preliminary)

- Finite, soluble, non-supersoluble, just-non-PT groups $\equiv$
  “standard type”: PT-group $\rtimes$ faithful simple module of dim $\geq 2$

- Finite, insoluble, PST, just-non-PT seem within reach
Application

- T-group: $H \trianglelefteq N \trianglelefteq G \implies H \trianglelefteq G$

- PST-group: $H \trianglelefteq N \trianglelefteq G \implies H \mbox{S-per } G$

- Soluble, PT-group: $H \trianglelefteq N \trianglelefteq G \implies H \mbox{ per } G$

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- Soluble, PT-group proof used soluble, PST, just-non-PT groups

- Ramon Esteban-Romero found a nice counterexample for **insoluble**
  PT-groups
An interesting example

- Just-non-PT, \( p \)-group \( P \) with \( M \trianglelefteq P \) such that
  \[ H \leq M, H \text{ per } P \iff H \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_k = P \]
  
  \[ P = \langle a, b, c : a^{pk+1} = b^p = c^p = [c, a] = 1, [a, b] = a^p, [c, b] = a^{p^k} \rangle, \]
  \[ M = \langle a, b \rangle \]

- \( |ZX| = |X/X'| = p \) and \( X'/ZX \) simple
  
  \[ X'/ZX = PSL(p, q) \text{ for } 1 \equiv q \mod p, X = \langle t, X' \rangle \]

- The central product of \( X \) and \( P \) contains a subgroup \( G = \langle X, M, tc^{-1} \rangle \)

- \( G \) is finite, insoluble, PST, just-non-PT, and every subnormal subgroup of defect at most \( k \) is permutable
Local classification

Define:

\[ N_p^o = \{ G : H \leq O_p(G) \implies [O^p(G), H] \leq H \} \]
\[ P_p^o = \{ G : H \leq O_p(G) \implies H \text{ per } G \} \]
\[ N_p^* = \{ G : G/M \in N_p^o, \forall M \trianglelefteq G \} \]
\[ P_p^* = \{ G : G/M \in P_p^o, \forall M \trianglelefteq G \} \]

Then:

\[ \text{PST} = N_p^* \text{ for all } p \text{ and simple chief factors} \]
\[ \text{PT} = \text{PST} \text{ and } P_p^* \text{ for all } p \]
\[ \text{PST, just-non-PT are just-non-} P_p^o \text{ for exactly one prime } p \]
\[ \text{and } P_q^* \text{ for all other primes } q \]
Lattice and partial results

Suppose: $G$ PST, just-non-PT group, $O_p$ non-abelian, $G_p$ a sylow of $G$, $D = O_\infty(G)$

- $G/D$ is soluble PT, so has standard form
- $G/DO_\infty \leq \text{Out}(D)$, so has standard form
- $O_p(G) = O_\infty(G)$, ”solvable part” is easy
- $D$ is central product of quasi-simples, $G_p$ acts diagonally on it
- $G$ itself should be a nice central product of just-non-PT with $D$ quasi-simple and PT
- $O^p(G) \neq O_\infty(G)$, at least in the “central irreducible” case?
Summary

- Just-non-\(X\) groups are interesting when \(X\) is interesting

- PT used to have the same classifications as T and PST, but now is special

- We should understand just-non-PT groups, especially insoluble

- We need both concrete and conceptual classifications

The End