Schedule:

- HW 2.3-2.4 are due Friday, Sep 16th, 2011.
- HW 2.5-2.6 are due Friday, Sep 23rd, 2011.
- Exam 1 is Monday, Sep 26th, 5:00pm-7:00pm in CB106.

Today we will cover 2.3 and pages 7-8 of the appendix: degeneracy and RREF.
Which section of road is busiest, x, y, or z?
Line-about: same thing with lines
Line-about: same thing with lines

\[ x + 4 = 3 + y \]
\[ y + 6 = 5 + z \]
\[ z + 4 = 6 + x \]
Line-about: same thing with lines

\[
\begin{align*}
x + 4 &= 3 + y \\
y + 6 &= 5 + z \\
z + 4 &= 6 + x \\
1x - 1y + 0z &= -1 \\
0x + 1y - 1z &= -1 \\
-1x + 0y + 1z &= 2
\end{align*}
\]
\[
x + 4 = 3 + y \\
y + 6 = 5 + z \\
z + 4 = 6 + x \\
1x + -1y + 0z = -1 \\
0x + 1y + -1z = -1 \\
-1x + 0y + 1z = 2 \\
\]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
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</table>
x + 4 = 3 + y
y + 6 = 5 + z
z + 4 = 6 + x

1x + -1y + 0z = -1
0x + 1y + -1z = -1
-1x + 0y + 1z = 2

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<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ R_3 + R_1 \]

<table>
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<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
We managed to solve some fairly big systems last time using our new number crunching skills.

Mostly it was repetitive, routine, soothing.

But near the end we stopped the number pushing and revived the variables, which totally harshed my zen.

Today we learn to finish the easy way
Appendix: Cleaning above as well as below

- A matrix is in **REF** if no column (left of the bar) has two pivots

- This means that below and to the left of each pivot are zeros

- A matrix is in **RREF** if
  - it is in REF,
  - there are only zeros above pivots, and
  - pivots are equal to 1
Appendix: How to clean

- If a matrix is in REF, then a **possible target** is a non-zero number above a pivot.

- We choose the right-most column with a possible target, and then choose the bottom-most possible target in that column.

- The row operation is the same as before:

  \[ R_{target} \rightarrow \frac{target}{active} \cdot R_{active} \]

- If a pivot is not equal to one, then we can divide the whole row by the pivot.
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\end{bmatrix}
\]
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5
\end{bmatrix}
\xrightarrow{R_2 - R_3}
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]
## Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
R_2 - R_3 
\rightarrow
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\] $\xrightarrow{R_2-R_3}$

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 & 0 & 10 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\begin{array}{c}
R_2 - R_3 \\
\rightarrow \\
R_1 - R_3 \\
\rightarrow \\
\end{array}
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 0 & 10 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & \text{15} \\
0 & 1 & 1 & \text{9} \\
0 & 0 & 1 & \text{5}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 1 & 1 & \text{15} \\
0 & 1 & 0 & \text{4} \\
0 & 0 & 1 & \text{5}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 1 & 0 & \text{10} \\
0 & 1 & 0 & \text{4} \\
0 & 0 & 1 & \text{5}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 0 & \text{6} \\
0 & 1 & 0 & \text{4} \\
0 & 0 & 1 & \text{5}
\end{bmatrix}
\]
## Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 1 & 9 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\xrightarrow{R_2 - R_3}
\begin{bmatrix}
2 & 1 & 1 & 15 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\xrightarrow{R_1 - R_3}
\begin{bmatrix}
2 & 1 & 0 & 10 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\xrightarrow{R_1 - R_2}
\begin{bmatrix}
2 & 0 & 0 & 6 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 5 \\
\end{bmatrix}
\]
Appendix: Example

\[ \begin{bmatrix} 2 & 1 & 1 & 15 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{bmatrix} \]

\[ R_2 - R_3 \quad \rightarrow \quad \begin{bmatrix} 2 & 1 & 1 & 15 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \]

\[ R_1 - R_3 \quad \rightarrow \quad \begin{bmatrix} 2 & 1 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \]

\[ R_1 - R_2 \quad \rightarrow \quad \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \]

\[ \frac{1}{2} R_1 \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix} \]
Appendix: Example

\[
\begin{bmatrix}
2 & 1 & 1 & | & 15 \\
0 & 1 & 1 & | & 9 \\
0 & 0 & 1 & | & 5 \\
\end{bmatrix}
\]

\[R_2 - R_3\]

\[
\begin{bmatrix}
2 & 1 & 1 & | & 15 \\
0 & 1 & 0 & | & 4 \\
0 & 0 & 1 & | & 5 \\
\end{bmatrix}
\]

\[R_1 - R_3\]

\[
\begin{bmatrix}
2 & 1 & 0 & | & 10 \\
0 & 1 & 0 & | & 4 \\
0 & 0 & 1 & | & 5 \\
\end{bmatrix}
\]

\[R_1 - R_2\]

\[
\begin{bmatrix}
2 & 0 & 0 & | & 6 \\
0 & 1 & 0 & | & 4 \\
0 & 0 & 1 & | & 5 \\
\end{bmatrix}
\]

\[\frac{1}{2} R_1\]

\[
\begin{bmatrix}
1 & 0 & 0 & | & 3 \\
0 & 1 & 0 & | & 4 \\
0 & 0 & 1 & | & 5 \\
\end{bmatrix}
\]

RREF
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Row 2 can only make row 1 worse and vice versa!
Row 3 cannot do anything at all!
Let's write it out in variables, and see what is going on:
\[x + y = 1\]
\[z = 0\]

Well that is not too bad?
\[x = 1\]
\[y, y\] is free,
\[z = 1.\]

We can read this right from the matrix

We do say this matrix is in REF and RREF
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

- What could we do to fix it?
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
2.3: What if things go wrong?

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- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
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2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 1 \\
0 & 0 & 0 & | & 0 \\
\end{bmatrix}
\]

- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
  - Row 3 cannot do anything at all!

- Let’s write it out in variables, and see what is going on:

\[
x + y = 1 \quad z = 1 \quad 0 = 0
\]
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0
\end{bmatrix}
\]

- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
  - Row 3 cannot do anything at all!

- Let's write it out in variables, and see what is going on:
  \[
  x + y = 1 \quad z = 1 \quad 0 = 0
  \]

- Well that is not too bad? \( x = 1 - y, \) \( y \) is free, \( z = 1. \)
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- What could we do to fix it?
  - Row 2 can only make row 1 worse and vice versa!
  - Row 3 cannot do anything at all!

- Let’s write it out in variables, and see what is going on:

\[
x + y = 1 \quad z = 1 \quad 0 = 0
\]

- Well that is not too bad? \( x = 1 - y \), \( y \) is free, \( z = 1 \).

- We can read this right from the matrix
2.3: What if things go wrong?

Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

What could we do to fix it?

- Row 2 can only make row 1 worse and vice versa!
- Row 3 cannot do anything at all!

Let’s write it out in variables, and see what is going on:

\[x + y = 1 \quad z = 1 \quad 0 = 0\]

Well that is not too bad? \(x = 1 - y\), \(y\) is free, \(z = 1\).

We can read this right from the matrix

We do say this matrix is in REF and RREF
2.3: Free variables

- If a column (for a variable) has no pivot, then that variable is free.

- Be careful when reading the answer off the matrix 110|1 means \( x + y = 1 \), so \( x = 1 - y \).

- If a variable is free, then (assuming there are any solutions) there are infinitely many solutions.

- What does “no solution” look like in matrix format?
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
  1 & 1 & 0 & 1 \\
  0 & 0 & 1 & 1 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

What could we do to fix it? Let's write it out in variables, and see what is going on:

\[\begin{align*}
x + y &= 1 \\
z &= 1 \\
0 &= 1
\end{align*}\]

Well that is not too bad? \[x = 1, \quad y, \text{ free}, \quad z = 1, \quad 0 = 1?\]

What?! No solution, inconsistent.

We can read this right from the matrix. We do say this matrix is in REF and RREF.
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

- What could we do to fix it?
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
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\end{bmatrix}
\]

- What could we do to fix it?

- Let's write it out in variables, and see what is going on:

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\begin{align*}
x + y &= 1 \\
z &= 1 \\
0 &= 1
\end{align*}
\]

Well that is not too bad? We can read this right from the matrix. We do say this matrix is in REF and RREF.
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 1 \\
0 & 0 & 0 & | & 1 \\
\end{bmatrix}
\]

- What could we do to fix it?

- Let's write it out in variables, and see what is going on:

\[
x + y = 1 \quad z = 1 \quad 0 = 1
\]

- Well that is not too bad? \( x = 1 - y \), \( y \) is free, \( z = 1 \), and \( 0 = 1 \)? What?!
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

- What could we do to fix it?

- Let's write it out in variables, and see what is going on:

\[
x + y = 1 \quad z = 1 \quad 0 = 1
\]

- Well that is not too bad? \( x = 1 - y \), \( y \) is free, \( z = 1 \), and \( 0 = 1 \)?

- What?!

- No solution, inconsistent
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

- What could we do to fix it?

- Let's write it out in variables, and see what is going on:

\[
x + y = 1 \quad z = 1 \quad 0 = 1
\]

- Well that is not too bad? \( x = 1 - y \), \( y \) is free, \( z = 1 \), and \( 0 = 1 \)?
  **What?!**

- **No solution, inconsistent**

- We can read this right from the matrix
2.3: What if things go wrong?

- Is this matrix in REF? RREF?

\[
\begin{bmatrix}
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\end{bmatrix}
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\[
x + y = 1 \quad z = 1 \quad 0 = 1
\]

- Well that is not too bad? \( x = 1 - y \), \( y \) is free, \( z = 1 \), and \( 0 = 1 \)?

  - **What?!**

- **No solution, inconsistent**

- We can read this right from the matrix

- We do say this matrix is in REF and RREF