Today we will cover 5.1: simple and compound interest.
We will be using calculators today.
Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations
5.1: Interest

- Businesses often need short-term use of expensive assets, so find renting attractive (often tax-deductible)

- Sometimes what a business needs most is just cash. In a small business, you don’t make money every day. A successful small business does make money, so can repay the money in the future.

- How can they rent cash? Why would somebody give them money today? For the promise of more money in the future. Interest

- How much more?
  - The more money being loaned, the more interest. Principal
  - The longer the money is loaned, the more interest. Time
5.1: Simple interest

For short term loans, people use a **simple** model for interest

\[ I = Prt \]

- There is the **Principal**, the amount of money borrowed, like $100
- There is a **rate** of interest, like 10% per year
- There is a **time** period, after which the money is due, like 1 year
- There is the **Interest**, the extra money due at the end,

\[ \text{like } (100) \cdot (10\% \text{ per year}) \cdot (1 \text{ year}) = 10. \]
5.1: Simple interest examples

\[ I = Prt \]

- If $100 is lent at 10% interest per year for six months, then
  \[ I = ($100) \cdot (10\% \text{ per year}) \cdot \left( \frac{1}{2} \text{ year} \right) = $5 \]

- If $100 is lent at 7% interest per year for three months, then
  \[ I = ($100) \cdot (7\% \text{ per year}) \cdot \left( \frac{1}{4} \text{ year} \right) = $1.75 \]

- If $325 is lent at 12% interest per year for five months, then
  \[ I = ($325) \cdot (12\% \text{ per year}) \cdot \left( \frac{5}{12} \text{ year} \right) = $16.25 \]
5.1: Consumer example

- My Brother-in-Law’s electricity bill came too soon one month
- Bill was $46.40 now, but $48.72 if 3 days late
- He didn’t have the money now, but would have it in a week (IRS refund)
- He did have a 48% APR credit card carrying a balance (4% interest per month)
- A pawn shop would loan him the money for one month 2% interest per month, $5 fee

Which is cheaper:

(L) Pay it late
(R) Put it on the credit card, and pay the credit card
(B) Pawn his watch for a month, then pay it back
5.1: Let's just see how much each costs

- (L) is easy: $48.72 total, $2.32 in interest

- (R) is easy: $46.40 plus 4% = $46.40(1.04) = $48.26

- (B) is easy: $46.40 plus 2% plus $5 = $46.40(1.02) + 5 = $52.33

- Decision is also easy: credit card is the cheapest

- If he had the money now, then cheapest was to pay it now $46.60

- There is a price to not having money
5.1: More examples

- What is the simple (yearly) interest rate if $100 is loaned for 3 months with $5 interest due?
  
  \[
  P = 100 \\
  t = 1/4 \text{ year} \\
  I = 5 \\
  r = ?
  \]
  
  \[
  I = Prt \\
  5 = (100)(r)(1/4) \\
  5 = 25r \\
  r = 1/5 = 20\%
  \]

- If the interest rate is 7% and $9.10 interest is due after three months, how much was loaned?
  
  \[
  r = 7\% \text{ per year} \\
  t = 1/4 \text{ year} \\
  I = 9.1 \\
  P = ?
  \]
  
  \[
  I = Prt \\
  9.1 = P(7\%)(1/4) \\
  36.4 = P(7\%) \\
  P = $36.4/7\% = 520
  \]
Simple interest treats loaned money like a loaned house.

Every month you pay money to borrow the house, and at the end of the year you give the house back.

The owner goes without his house for a year, but receives money in exchange (rent). The lender goes without his money for a year, but receives money in exchange (interest).

How much rent do you pay total? If each month you pay $300, then by the end of the year, you’ve paid $(12)(\$300) = \$3600$.

This calculation is similar to the interest calculations we just did. Each month you pay $(\$100) \cdot (12\% \text{ per year})(\frac{1}{12} \text{ year}) = \$1$ interest, and at the end of the year that is $(12)(\$1) = \$12$, or 12% of the original loan.
5.1: Got no money to pay the rent

- What happens when you cannot pay the rent? Well, typically lots of bad things.

- What if you borrow the rent from your friend? Maybe he charges interest too.

- What if you can’t pay back your friend? Maybe you borrow from another friend, and maybe they charge interest too.

- Interest on interest is called **compound interest**

- In finance and economics, nearly all interest is compound

- Simple interest is used for short-term loans
Harry, Gary, and Scary run a limitedly legal company specializing in short term loans.

In January, Bob borrowed $100 from Harry for one month at 10% per month interest.

In February, Gary stopped by to say hello, and that his brother was anxious for his $110. Bob didn’t have the $110, but Gary said he looked like a nice guy and would loan him the $110 at 10% per month interest.

Bob asked if Gary minded giving Harry the money, since they were brothers, and so Gary took back the money immediately and went back to the cement yard.
In March, Scary stopped by to say hello, and that his brother was anxious for his $121

Bob didn’t have the $120, wait, $121?

If Bob had borrowed $100 for 2 months at 10% per month interest, then he would owe:

\[ 100 + (100) \cdot (10\% \text{ per month}) \cdot (2 \text{ months}) = 100 + 20 \]

However, he had borrowed $110 from Gary for 1 month at 10% per month interest, so he owed:

\[ 110 + (110) \cdot (10\% \text{ per month}) \cdot (1 \text{ month}) = 110 + 11 \]

Scary had no interest in the math, only in the interest, $10 from the first month, $11 from the second
The most basic formula for compound interest is:

\[ A = P(1 + i)^n \]

- The **Principal** is the amount initially borrowed, like $100.
- The **interest rate** per compounding period, like 10% per month.
- The **number** of compounding periods that have passed, like 2 months.
- The **Accumulated Amount** of money due, both the principal and the interest, like

\[
($100)(1 + 10\%)^2 = ($100)(1.10)^2 = $121
\]
5.1: Compound interest examples

\[ A = P(1 + i)^n \]

- If you borrow $100 at 10% per month, compounded monthly, for six months you owe
  \[ ($100) \cdot (1.1)^6 \approx $177.16 \]

- If you borrow $100 at 10% per month, compounded monthly, for nine months you owe
  \[ ($100) \cdot (1.1)^9 \approx $235.79 \]

- If you borrow $100 at 10% per month, compounded monthly, for twelve months you owe
  \[ ($100) \cdot (1.1)^{12} \approx $313.84 \]
5.1: Why does the formula work?

- If you borrow $100 at 10% per month, compounded monthly, for one month you owe

  \[ 100 + (100)(10\%) = 100 \cdot (1 + 10\%) = 100 \cdot (1.1) = 110 \]

- If you borrow it for another month, you owe

  \[ 110 + (110)(10\%) = (110)(1 + 10\%) = (110)(1.1) \]
  \[ = (100)(1.1)(1.1) = (100)(1.1)^2 = 121 \]

- If you borrow it for another month, you owe

  \[ 121 + (121)(10\%) = (121)(1 + 10\%) = (121)(1.1) \]
  \[ = (100)(1.1)^2(1.1) = (100)(1.1)^3 = 133.10 \]
5.1: Confusing customers for fun and profit

- Stating interests rates in terms of months, fortnights, or furlongs makes it hard to compare interest rates.

- A simple way to handle this is to multiply the rate by how many periods there are per year, to “convert” to a yearly rate, like 
  \[(10\% \text{ per month}) \cdot (12 \text{ months per year}) = 120\% \text{ per year}\]

- The **nominal rate** is this rate, “120% interest per year, compounded monthly.”

- To convert from a nominal rate to a per-period rate just divide by the number of periods.

- a nominal rate of 12% per year compounded monthly is a rate of 
  \[(12\% \text{ per year})/(12 \text{ months per year}) = 1\% \text{ per month}\]
However, what happens to Bob (best-case scenario) if he continues to get loans from the three brothers?

The nominal rate was 120% per year, compounded monthly

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\[ \text{Jan} \quad \text{value} = \text{initial value} + \text{interest} \]

\[ \text{Jan} \quad 313.84 = 100 + 213.84 \]

“120% per year compounded monthly” fails to capture the “213.84% per year simple interest”
5.1: Effective interest rate

- In the U.S. the 1968 Truth in Lending Act required lenders to advertise the **effective** annual percentage rate.

- The true calculation is complicated, depends on your jurisdiction, and takes into account certain fees and penalties.

- In MA162, the formula is not so complicated. You just calculate the interest for one year.

- For instance, the three brothers nominal rate of 120% resulted in

\[
(1 + \frac{1.20}{12})^{12} - 1 = (1 + 0.10)^{12} - 1 = 1.1^{12} - 1 \approx 2.13843 = 213.843\%
\]

- In general

\[
r_{\text{eff}} = (1 + \frac{r}{m})^m - 1
\]
5.1: Summary

- Today we learned **simple interest**, **compound interest**, and the **effective interest rate**.

- We used the words **interest**, **principal**, **interest rate**, **compounding period**, **nominal rate**, **accumulated amount**.

- You are now ready to complete HW 5.1

- Make sure to take advantage of office hours, and have your questions ready for your next recitation.