Schedule:

- Exam 2 returned during recitation
- HW 0.1-4.1 extended to Tue, March 20th, 2012
- HW 5.1, 5.2 are due Fri, March 23rd, 2012
- HW 5.3, 6.1 are due Fri, March 30th, 2012
- HW 6.2, 6.3 are due Fri, April 6th, 2012
- Exam 3 is Monday, Apr 9th, 5:00pm-7:00pm in CB106 and CB118.

Today we will cover 5.2: annuities.
Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
  - Compound interest
  - Sinking funds
  - Amortized loans

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations
“Annuity” can refer to a wide variety of financial instruments, often associated with retirement.

For us: it is a steady flow of cash into an interest bearing account.

For instance, “$100 invested at the end of every month, earning 1% per month compound interest at the end of every month (12% APR), is worth $1200+$68.25 at the end of the year.”

The $1200 part is just the 12 payments of $100.

How do we figure out the “+$68.25” part?
5.2: Spreadsheet method for annuity

- Four columns: Old balance, Interest, Payment, New Balance

<table>
<thead>
<tr>
<th>Date</th>
<th>Old</th>
<th>Int</th>
<th>Pay</th>
<th>New</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>$0.00</td>
<td>$0.00</td>
<td>$100.00</td>
<td>$100.00</td>
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<td>$1.00</td>
<td>$100.00</td>
<td>$201.00</td>
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<td>Mar</td>
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<td>$100.00</td>
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<td>Apr</td>
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<td>$4.06</td>
<td>$100.00</td>
<td>$510.10</td>
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<td>Oct</td>
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<td>$9.37</td>
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<td>Nov</td>
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<td>$10.46</td>
<td>$100.00</td>
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<tr>
<td>Dec</td>
<td>$1156.68</td>
<td>$11.57</td>
<td>$100.00</td>
<td>$1268.25</td>
</tr>
</tbody>
</table>
5.2: Formula method

\[ A = R((1 + i)^n - 1)/i \]

- where the **Recurring payment** is how much is deposited at the end of each period, like $100

- the **interest rate** per period, like 1%/12

- the **number of periods**, like four months

- the **accumulated amount**, like

\[ A = 100 \times ((1 + 0.01)^{12} - 1)/(0.01) = 1268.25 \]

\[ A = 100 \times ((1 + 0.01)^{12} - 1)/(0.01) = 1268.250301 \]
5.2: Examples of formula

\[ A = R((1 + i)^n - 1)/i \]

- After one year of investing $100 at the end of every month at a 1\% \text{ (nominal yearly)} interest rate:
  \[ R = $100 \]
  \[ i = 1\%/12 \approx 0.00833333 \]
  \[ n = 12 \text{ months} \]
  \[ A = $100((1 + 1\%/12)^{12} - 1)/(1%/12) \approx $1205.52 \]

- After two years of investing $100 at the end of every month at a 1\% \text{ (nominal yearly)} interest rate:
  \[ R = $100 \]
  \[ i = 1\%/12 \approx 0.00833333 \]
  \[ n = 24 \text{ months} \]
  \[ A = $100((1 + 1\%/12)^{24} - 1)/(1%/12) \approx $2423.14 \]
5.2: Retirement example

- UK employees aged 30 or over must contribute 5% of their salary each month to a retirement plan, which UK doubles, a total of 15%

- If a UK employee makes $35k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:
  \[ R = \left(\$35000\right)\left(15\%\right)/12 = \$437.50 \]
  \[ i = 8%/12 \]
  \[ n = (35)(12) = 420 \text{ months} \]
  \[ A = \$437.50\left(\left(1 + \frac{8\%}{12}\right)^{420} - 1\right)/\left(\frac{8\%}{12}\right) \approx \$1,003,573.59 \]

- If a UK employee makes $70k and retires at age 65 and manages to earn a steady 8% interest rate, then they retire with:
  \[ R = \$875 \]
  \[ i = 8%/12 \]
  \[ n = (35)(12) = 420 \text{ months} \]
  \[ A = \$875\left(\left(1 + \frac{8\%}{12}\right)^{420} - 1\right)/\left(\frac{8\%}{12}\right) \approx \$2,007,147.18 \]
5.2: Sinking fund example

- Businesses can often predict future expenses; our building needs a new water boiler ($80k) after this one breaks.

- We set aside a little each month so that we have it when we need it.

- If we can get 3% interest in low-risk investments and expect the boiler to fail in 5 years, we need to invest $R$ per month:

\[
A = \frac{R((1 + i)^n - 1)}{(i)}
\]

\[
R = ?
\]

\[
i = \frac{3\%}{12}
\]

\[
n = (5)(12) = 60 \text{ months}
\]

\[
A = $80000
\]

\[
\frac{$80000}{(1 + \frac{3\%}{12})^{60} - 1}/(\frac{3\%}{12}) = R(64.64671280)
\]

\[
R = \frac{$80000}{64.64671280} = $1237.50
\]
5.2: Sinking fund versus one-time-investment

- Maybe we don’t want to pay a little each month

- Maybe we just want to invest a whole bunch now and cash in later
  \[ A = P(1 + i)^n \]
  
  \[ P = ? \]
  
  \[ i = 3\%/12 \]
  
  \[ n = (5)(12) = 60 \text{ months} \]
  
  \[ A = $80000 \]
  
  \[ $8000 = P(1 + 3\%/12)^{60} \]
  
  \[ $8000 = P(1.161616782) \]
  
  \[ P = $80000/1.161616782 = $68869.53 \]

- Less total money we invested for same future value

- But we need that $68k NOW, not $1.2k at a time
5.2: Why does the formula work?

- After one month you have $100

- The next month you add a fresh $100 and \((1+i)\) times your previous month
  
  \[100 + 100 \cdot (1 + i)\]

- The next month you add a fresh $100 and \((1+i)\) times your previous month
  
  \[100 + (100 + 100 \cdot (1 + i)) \cdot (1 + i)\]

  \[100 + 100 \cdot (1 + i) + 100 \cdot (1 + i)^2\]

- The next month you add a fresh $100 and \((1+i)\) times your previous month
  
  \[100 + (100 + (100 + 100 \cdot (1 + i)) \cdot (1 + i)) \cdot (1 + i)\]

  \[100 + 100 \cdot (1 + i) + 100 \cdot (1 + i)^2 + 100 \cdot (1 + i)^3\]
5.2: Trick for summations

- After $n$ months you have added up $n$ things:

$$A = 100 + 100 \cdot (1 + i) + \cdots + 100 \cdot (1 + i)^{n-1}$$

- Let's try a trick. What happens if I let the money ride for a month? It earns interest, so I have $A \cdot (1 + i)$ in the bank.

- How much more is that? Well $A \cdot (1 + i) - A = Ai$ is not tricky.

- But multiply it out before doing the subtraction:

$$\begin{align*}
A \cdot (1 + i) &= 100 + 100 \cdot (1 + i) + \cdots + 100 \cdot (1 + i)^{n-1} + 100 \cdot (1 + i)^n \\
- A &= -100 + 100 \cdot (1 + i) + \cdots + 100 \cdot (1 + i)^{n-1} + 100 \cdot (1 + i)^n \\
Ai &= -100
\end{align*}$$

- So $Ai = 100 \cdot ((1 + i)^n - 1)$ and we can solve for $A$:

$$A = 100 \frac{(1 + i)^n - 1}{i}$$
How much would you pay me for (the promise of) $100 in a year?

Future money is not worth as much as money right now

“A bird in the hand, is worth two in the bush” posits an interest rate of 100%

Present value of future money **depreciates** the value of future money by comparing it to present money invested in the bank now.

**Total payout** is a popular measure of a financial instrument, but it mixes present money, with in-a-little-while money, with future money.

Total payout of an annuity is just the total amount you put in the savings account (or the total amount you borrowed each month).
5.2: Summary

- Today we learned about **annuities, present value, future value**, and **total payout**

- Future value of annuity, paying out $n$ times at per-period interest rate $i$

  \[ A = R \frac{(1 + i)^n - 1}{i} \]

- Present value of annuity is just future value divided by $(1 + i)^n$

- Total payout is just $nR$, $n$ payments of $R$ each

- You are now ready to complete HW 5.2 and should have already completed HW 5.1

- Make sure to take advantage of office hours: today 2pm-3pm in Mathskeller (CB63, basement of White Hall Classroom Building)
How much would you pay today for an annuity paying you back $100 per month for 12 months?

No more than $1200 for sure, if you had $1200 you could just pay yourself

Let’s try to find the right price for such a cash flow

What if you didn’t need the money? You could deposit it each month into your savings account.

We already calculated that you end up with $1205.52 if you do that

How much would you pay today for $1205.52 in the bank a year from now?
If you had $1193.53 and just put it in the bank now, you’d end up with $1193.53 \left(1 + \frac{1}{12}\right)^{12} = $1205.52 anyways.

If you were just concerned with how much you had in the bank at the end, then you would have no preference between $1193.53 up front and $100 each month.

In other words, the \textbf{present value} of the $100 each month for a year is $1193.53 because both of those have the same \textbf{future value}.

What if you do need the money each month? Is $1193.53 still the right price?
5.3: Pricing annuities again

- What would happen if you put $1193.53 in the bank, and withdrew $100 each month?

- At the end of the year, you’d have $0.00 in the bank, but you would not be overdrawn.

- Why is that? Imagine borrowing money from your friend, $100 every month and not paying them back

- They know you pretty well, so they insisted on 1% interest, compounded monthly

- How much do you owe them at the end?

- Well from their point of view, they gave their money to you, just like putting it in a savings account

- The bank would have owed them $1205.52, so you owe them $1205.52. Now imagine your savings account is your friend.