Exam 2 returned during recitation
HW 5.1,5.2 are due Fri, March 23rd, 2012
HW 5.3,6.1 are due Fri, March 30th, 2012
HW 6.2,6.3 are due Fri, April 6th, 2012
Exam 3 is Monday, Apr 9th, 5:00pm-7:00pm in CB106 and CB118.

Today we will cover 5.3: amortized loans.

We will be using calculators today.
Exam 3 breakdown

- Chapter 5, Interest and the Time Value of Money
  - Simple interest
    - short term, interest not reinvested
  - Compound interest
    - one payment, interest reinvested
  - Sinking funds
    - recurring payments, big money in the future
  - Amortized loans
    - recurring payments, big money in the present

- Chapter 6, Counting
  - Inclusion exclusion
  - Inclusion exclusion
  - Multiplication principle
  - Permutations and combinations
Monday we learned about **annuities, present value, future value, and total payout**

- Future value of annuity, paying out \( n \) times at per-period interest rate \( i \)

\[
A = R \frac{(1 + i)^n - 1}{i}
\]

- Present value of annuity is just future value divided by \((1 + i)^n\)

- Total payout is just \( nR \), \( n \) payments of \( R \) each

You should be done with homework for 5.1 and 5.2.

Today we handle 5.3.
5.3: Buying annuities

- How much would you pay today for an annuity paying you back $100 per month for 12 months?

- No more than $1200 for sure, if you had $1200 you could just pay yourself.

- If you have a 12% APR (1% per month) account, then you could invest the money each month, in one year you have $1268.25.

- How much would you need right now (one payment) in order to have $1268.25 in the account after one year?
We solve a 5.1 problem:

\[ P = ? \]
\[ i = \frac{0.12}{12} = 0.01 \text{ per month} \]
\[ n = 12 \text{ months} \]
\[ A = $1268.25 \]
\[ A = P(1 + i)^n \]
\[ $1268.25 = P(1.01)^{12} \]
\[ P = \frac{$1268.25}{(1.01)^{12}} = $1125.50 \]

- If we had $1125.50 right now, we could invest it to end up with $1268.25
- If we got $100 every month, we could invest it to end up with $1268.25
- So the cash flow is worth $1125.50 now
5.3: Pricing annuities again

- What if we don’t want to invest it?
  - What if we want to spend $100 every month?

- Well, put $1125.50 in the bank and remove $100 every month

- How much is left at the end of the year?

<table>
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<tr>
<th>Date</th>
<th>Old Balance</th>
<th>Interest on Old</th>
<th>Withdrawal</th>
<th>New Balance</th>
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<tr>
<td>Dec</td>
<td>$99.00</td>
<td>$0.99</td>
<td>$100.00</td>
<td>$-0.01</td>
</tr>
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</table>
5.3: Pricing an annuity

- To price an annuity using our old formulas:

  - Find the future value \( A = R((1 + i)^n - 1)/(i) \)

  - Find the present value by solving \( A = P(1 + i)^n \)

    \[
    P = A/((1 + i)^n)
    \]

- If you like new formulas, the book divides the \((1 + i)^n\) using algebra:

  \[
  P = R \left(1 - (1 + i)^{(-n)}\right) /(i)
  \]
5.3: Perspective

- Alex borrows $100 per month from Bart at 1% per month interest, compounded monthly.

- Bart thinks of Alex as a savings account.

- Bart expects $1268.25 in his account at the end of the year.

- Alex owes Bart $1268.25 at the end of the year.

  What if the bank called you up and wanted to buy an annuity?

- What if Bart wants Alex to pay in advance? How much does Alex owe Bart up front?
5.3: Amortized loan

- Most people don’t say “the bank purchased an annuity from me”
- “I owe the bank money every month, because they gave me a loan”
- So the bank gives you $1125.50 and expects 1% interest per month
- You give the bank $100 back at the end of the month

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<td>$ 0.99</td>
<td>$100</td>
<td>$ -0.01</td>
</tr>
</tbody>
</table>

- You owe:
  - $1125.50 + (1% of it) - $100
  - $1125.50 + $11.26 - $100
  - $1036.76

- Amortized loans are just present values of annuities
5.3: Finding the time

If you owe $1000 at 12% interest compounded monthly and pay back $20 per month, how long does it take to pay it off?

After one month, you owe $1000 + $10 interest - $20 payment, a total of $990. So each month the debt goes down by a net $10. Should take 99 more months, or a little more than 8 years.

After two months, you owe $990 + $9.90 interest - $20 payment, a total of $979.90. Now it went down by $10.10! Should take $979.90/$10.10 = 97 months.

After one month of paying, we estimate two months fewer. How many is it really?
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How many is it really?
5.3: Finding the time

- The debt is paid once the future value of the annuity is equal to the future value of the debt

- **Annuity:**
  \[ A = R \frac{(1 + i)^n - 1}{i} \]
  
  \[ R = \$20 \]
  \[ i = \frac{0.12}{12} = 0.01 \]
  \[ n = ? \]
  \[ A = \ldots \]

- **Debt:**
  \[ A = P(1 + i)^n \]
  
  \[ P = \$1000 \]
  \[ i = 0.01 \]
  \[ n = ? \]
  \[ A = \$1000(1.01)^n \]

- So solve:
  \[ \frac{20(1.01^n - 1)}{0.01} = $1000(1.01)^n \]
Need to solve:

\[ 20(1.01^n - 1)/0.01 = 1000(1.01)^n \]

divide both sides by $1000$ and notice $20/0.01/1000 = 2$:

\[ 2(1.01^n - 1) = 1.01^n \]

distribute:

\[ 2(1.01^n) - 2 = 1.01^n \]

subtract $1.01^n$ from both sides, add 2 to both sides:

\[ 1.01^n = 2 \]

Now what?
To solve:

\[ 1.01^n = 2 \]

Take **logarithms** of both sides:

\[ (n)(\log(1.01)) = \log(2) \]

\log(1.01) is just a number (some might say 0.004321373783)

Divide both sides by \log(1.01) to get:

\[ n = \frac{\log(2)}{\log(1.01)} \approx 69.66 \approx 70 \]

\[ n = 70 \text{ months} \]

Monthly payments are worth the same as the debt after 70 months