Chapter 1. Introduction

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The widespread availability of personal computers has brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple, Mathematica, and MATLAB, each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. Maple and Mathematica also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to delegate as many as possible of the routine (often repetitive) details to a computer, while you focus on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require a calculator or computer. Good judgment is often needed in selecting a judicious combination.

PROBLEMS
In each of Problems 1 through 6 determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. \( r^2 \frac{d^2y}{dt^2} + t \frac{dy}{dt} + 2y = \sin t \)
2. \( (1 + y^2) \frac{d^2y}{dt^2} + t \frac{dy}{dt} + y = e^t \)
3. \( \frac{d^2y}{dt^2} + \frac{dy}{dt^2} + \frac{dy}{dt} + y = 1 \)
4. \( \frac{dy}{dt} + ty^2 = 0 \)
5. \( \frac{d^2y}{dt^2} + \sin(t + y) = \sin t \)
6. \( \frac{d^2y}{dt^2} + t \frac{dy}{dt} + (\cos^2 t)y = t^3 \)
In each of Problems 7 through 14 verify that each given function is a solution of the differential equation.

7. \( y'' - y = 0; \quad y_1(t) = e^t, \quad y_2(t) = \cosh t \)
8. \( y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-t}, \quad y_2(t) = e^{3t} \)
9. \( y'' - y = r^2; \quad y = 2t + 2 \)
10. \( y'' + 4y'' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3 \)
11. \( 2r^2y'' + 3ry' - y = 0, \quad t > 0; \quad y_1(t) = t^{2/3}, \quad y_2(t) = t^{-1} \)
12. \( r^2y'' + 5ry' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = -t^{-2} \ln t \)
13. \( y'' + y = \sec t, \quad 0 < t < \pi/2; \quad y = (\cos t) \ln \cos t + t \sin t \)
14. \( y' - 2ty = 1; \quad y = e^t \int_0^t e^{-s} ds + e^t \)

In each of Problems 15 through 18 determine the values of \( r \) for which the given differential equation has solutions of the form \( y = e^r \).

15. \( y' + 2y = 0 \)
16. \( y'' - y = 0 \)
17. \( y'' + y' - 6y = 0 \)
18. \( y'' - 3y' + 2y = 0 \)

In each of Problems 19 and 20 determine the values of \( r \) for which the given differential equation has solutions of the form \( y = e^{rt} \) for \( t > 0 \).

19. \( r^2y'' + 4ry' + 2y = 0 \)
20. \( 2r^2y'' - 4ry' + 4y = 0 \)

In each of Problems 21 through 24 determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

21. \( u_{xx} + u_{yy} + u_{zz} = 0 \)
22. \( u_{xx} + u_{xy} + u_{yx} + u_{yy} + u = 0 \)
23. \( u_{xxx} + 2u_{xxy} + u_{yyy} = 0 \)
24. \( u_t + uu_x = 1 + u_{xx} \)

In each of Problems 25 through 28 verify that each given function is a solution of the given partial differential equation.

25. \( u_{xx} + u_{yy} = 0; \quad u_1(x, y) = \cos x \cosh y, \quad u_2(x, y) = \ln(x^2 + y^2) \)
26. \( \alpha^2 u_{xx} = u_t; \quad u_1(x, t) = e^{-\alpha^2 t} \sin \alpha x, \quad u_2(x, t) = e^{-\alpha^2 t} \sin \alpha x, \quad \alpha \) a real constant
27. \( \alpha^2 u_{xx} = u_t; \quad u_1(x, t) = \sin \alpha x \sin \alpha t, \quad u_2(x, t) = \sin(x - \alpha t), \quad \alpha \) a real constant
28. \( \alpha^2 u_{xx} = u_t; \quad u = (\pi \beta)^{1/2} e^{-t/\beta^2}, \quad t > 0 \)

29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.

(a) Assume that the mass is in an arbitrary displaced position, indicated by the angle \( \theta \). Draw a free-body diagram showing the forces acting on the mass.
(b) Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is \( Ld^2\theta/dt^2 \), where \( L \) is the length of the rod.
(c) Simplify the result from part (b) to obtain Eq. (12) in the text.

30. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

(a) Show that the kinetic energy \( T \) of the pendulum in motion is

\[
T = \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2 .
\]
2.1 Linear Equations; Method of Integrating Factors

in Chapter 8, that proceed directly from the differential equation and need no expression for the solution. Software packages such as Maple and Mathematica readily execute such procedures and produce graphs of solutions of differential equations.

![Graph of integral curves](image)

**FIGURE 2.1.4** Integral curves of \(2y' + ty = 2\).

Figure 2.1.4 displays graphs of the solution (47) for several values of \(c\). From the figure it may be plausible to conjecture that all solutions approach a limit as \(t \to \infty\). The limit can be found analytically (see Problem 32).

**PROBLEMS**

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.
(b) Based on an inspection of the direction field, describe how solutions behave for large \(t\).
(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as \(t \to \infty\).

\[\begin{align*}
1. \ y' + 3y &= t + e^{-2t} \\
2. \ y' - 2y &= r^2 e^{at} \\
3. \ y' + y &= te^{-t} + 1 \\
4. \ y' + (1/t)y &= 3 \cos 2t, \quad t > 0 \\
5. \ y' - 2y &= 3e^t \\
6. \ ty' + 2y &= \sin t, \quad t > 0 \\
7. \ y' + 2ty &= 2te^{-t^2} \\
8. \ (1 + r^2)y' + 2ty = (1 + r^2)^2 \\
9. \ 2y' + y &= 3t \\
10. \ ty' - y &= r^2 e^{-t}, \quad t > 0 \\
11. \ y' + y &= 5 \sin 2t \\
12. \ 2y' + y &= 3t^3
\end{align*}\]

In each of Problems 13 through 20 find the solution of the given initial value problem.

\[\begin{align*}
13. \ y' - y &= 2te^2, \quad y(0) = 1 \\
14. \ y' + 2y &= te^{-2t}, \quad y(1) = 0 \\
15. \ ty' + 2y &= r^2 - t + 1, \quad y(1) = 1/3, \quad t > 0 \\
16. \ y' + (2/t)y &= (\cos t)/t^2, \quad y(\pi) = 0, \quad t > 0 \\
17. \ y' - 2y &= e^{2t}, \quad y(0) = 2
\end{align*}\]
Chapter 2. First Order Differential Equations

18. \( ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0 \)
19. \( t^2y' + 4ty = e^{-t}, \quad y(-1) = 0, \quad t < 0 \)
20. \( ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0 \)

In each of Problems 21 through 23:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as \( t \) becomes large? Does the behavior depend on the choice of the initial value \( a \)? Let \( a_0 \) be the value of \( a \) for which the transition from one type of behavior to another occurs. Estimate the value of \( a_0 \).
(b) Solve the initial value problem and find the critical value \( a_0 \) exactly.
(c) Describe the behavior of the solution corresponding to the initial value \( a_0 \).

21. \( y' - \frac{1}{2}y = 2\cos t, \quad y(0) = a \)
22. \( 2y' - y = e^{t^2}, \quad y(0) = a \)
23. \( 3y' - 2y = e^{-\sqrt{t}}, \quad y(0) = a \)

In each of Problems 24 through 26:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as \( t \to 0^+ \)? Does the behavior depend on the choice of the initial value \( a \)? Let \( a_0 \) be the value of \( a \) for which the transition from one type of behavior to another occurs. Estimate the value of \( a_0 \).
(b) Solve the initial value problem and find the critical value \( a_0 \) exactly.
(c) Describe the behavior of the solution corresponding to the initial value \( a_0 \).

24. \( ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0 \)
25. \( ty' + 3y = (\sin t)/t, \quad y(-\pi/2) = a, \quad t < 0 \)
26. \( (\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi \)
27. Consider the initial value problem
\[
y' + \frac{1}{2}y = 2\cos t, \quad y(0) = -1.
\]
Find the coordinates of the first local maximum point of the solution for \( t > 0 \).

28. Consider the initial value problem
\[
y' + \frac{1}{2}y = 1 - \frac{1}{2}t, \quad y(0) = y_0.
\]
Find the value of \( y_0 \) for which the solution touches, but does not cross, the \( t \)-axis.

29. Consider the initial value problem
\[
y' + \frac{1}{2}y = 3 + 2\cos 2t, \quad y(0) = 0.
\]
(a) Find the solution of this initial value problem and describe its behavior for large \( t \).
(b) Determine the value of \( t \) for which the solution first intersects the line \( y = 12 \).

30. Find the value of \( y_0 \) for which the solution of the initial value problem
\[
y' - y = 1 + 3\sin t, \quad y(0) = y_0
\]
remains finite as \( t \to \infty \).

31. Consider the initial value problem
\[
y' - \frac{1}{3}y = 3t + 2e^t, \quad y(0) = y_0.
\]
2.1 Linear Equations; Method of Integrating Factors

Find the value of $y_0$ that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of $y_0$ behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
   \text{Hint: Consider the general solution, Eq. (47), and use L'Hopital's rule on the first term.}

33. Show that if $a$ and $\lambda$ are positive constants, and $b$ is any real number, then every solution of the equation
   \[ y' + ay = be^{-\lambda t} \]
   has the property that $y \to 0$ as $t \to \infty$.
   \text{Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.}

In each of Problems 34 through 37 construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.
35. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.
36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.
37. All solutions approach the curve $y = 4 - r^2$ as $t \to \infty$.
38. Variation of Parameters. Consider the following method of solving the general linear equation of first order:

   \[ y' + p(t)y = g(t). \] \hspace{1cm} (i)

   (a) If $g(t) = 0$ for all $t$, show that the solution is

   \[ y = A \exp \left[ - \int p(t) \, dt \right], \] \hspace{1cm} (ii)

   where $A$ is a constant.

   (b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

   \[ y = A(t) \exp \left[ - \int p(t) \, dt \right], \] \hspace{1cm} (iii)

   where $A$ is now a function of $t$. By substituting for $y$ in the given differential equation, show that $A(t)$ must satisfy the condition

   \[ A'(t) = g(t) \exp \left[ \int p(t) \, dt \right]. \] \hspace{1cm} (iv)

   (c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine $y$. Verify that the solution obtained in this manner agrees with that of Eq. (33) in the text. This technique is known as the method of \textit{variation of parameters}; it is discussed in detail in Section 3.6 in connection with second order linear equations.

In each of Problems 39 through 42 use the method of Problem 38 to solve the given differential equation.

39. $y' - 2y = t^2 e^t$

40. $y' + \left(1/t\right)y = 3 \cos 2t, \quad t > 0$

41. $ty' + 2y = \sin t, \quad t > 0$

42. $2y' + y = 3t^2$
we see that the interval ends when we reach points where the tangent line is vertical. It follows from the differential equation (22) that these are points where \( 4 + y^3 = 0 \), or \( y = (-4)^{1/3} \approx -1.5874 \). From Eq. (24) the corresponding values of \( x \) are \( x \approx \pm 3.3488 \). These points are marked on the graph in Figure 2.2.3.

**Note 1:** Sometimes an equation of the form (2)

\[
\frac{dy}{dx} = f(x, y)
\]

has a constant solution \( y = y_0 \). Such a solution is usually easy to find because if \( f(x, y_0) = 0 \) for some value \( y_0 \) and for all \( x \), then the constant function \( y = y_0 \) is a solution of the differential equation (2). For example, the equation

\[
\frac{dy}{dx} = \frac{(y - 3) \cos x}{1 + 2y^2}
\]  \hspace{1cm} (25)

has the constant solution \( y = 3 \). Other solutions of this equation can be found by separating the variables and integrating.

**Note 2:** The investigation of a first order nonlinear equation can sometimes be facilitated by regarding both \( x \) and \( y \) as functions of a third variable \( t \). Then

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dx}{dt}.
\]  \hspace{1cm} (26)

If the differential equation is

\[
\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)},
\]  \hspace{1cm} (27)

then, by comparing numerators and denominators in Eqs. (26) and (27), we obtain the system

\[
dx/dt = G(x, y), \quad dy/dt = F(x, y).
\]  \hspace{1cm} (28)

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but, in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

**Note 3:** In Example 2 it was not difficult to solve explicitly for \( y \) as a function of \( x \). However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 1 and 3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

**PROBLEMS**

In each of Problems 1 through 8 solve the given differential equation.

1. \( y' = x^2/y \)
2. \( y' = x^2/y(1 + x^3) \)
3. \( y' + y^2 \sin x = 0 \)
4. \( y' = (3x^2 - 1)/(3 + 2y) \)
5. \( y' = (\cos^2 x)(\cos^2 2y) \)
6. \( xy' = (1 - y^2)^{1/2} \)
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7. \( \frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y} \)

8. \( \frac{dy}{dx} = \frac{x^2}{1 + y^2} \)

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.
(b) Plot the graph of the solution.
(c) Determine (at least approximately) the interval in which the solution is defined.

9. \( y' = (1 - 2x)y^2, \quad y(0) = -1/6 \)
10. \( y' = (1 - 2x)/y, \quad y(1) = -2 \)
11. \( x \, dx + ye^{-x} \, dy = 0, \quad y(0) = 1 \)
12. \( dr/d\theta = r^2/\theta, \quad r(1) = 2 \)
13. \( y' = 2x/(y + x^2), \quad y(0) = -2 \)
14. \( y' = x/y^2(1 + x^2)^{-1/2}, \quad y(0) = 1 \)
15. \( y' = 2x/(1 + 2y), \quad y(2) = 0 \)
16. \( y' = x/(x^2 + 1)/4y^3, \quad y(0) = -1/\sqrt{2} \)
17. \( y' = (3x^2 - e^y)/(2y - 10), \quad y(0) = 1 \)
18. \( y' = (e^{-y} - e^y)/(3 + 4y), \quad y(0) = 1 \)
19. \( \sin 2x \, dx + \cos 3y \, dy = 0, \quad y(\pi/2) = \pi/3 \)
20. \( y^2(1 - x^2)^{1/3} \, dy = \arcsin x \, dx, \quad y(0) = 1 \)

Some of the results requested in Problems 21 through 28 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion as to the advantages and disadvantages of each approach.

21. Solve the initial value problem

\[ y' = (1 + 3x^2)/(3y^2 - 6y), \quad y(0) = 1 \]

and determine the interval in which the solution is valid.

*Hint:* To find the interval of definition, look for points where the integral curve has a vertical tangent.

22. Solve the initial value problem

\[ y' = 3x^2/(3y^2 - 4), \quad y(1) = 0 \]

and determine the interval in which the solution is valid.

*Hint:* To find the interval of definition, look for points where the integral curve has a vertical tangent.

23. Solve the initial value problem

\[ y' = 2y^2 + xy^2, \quad y(0) = 1 \]

and determine where the solution attains its minimum value.

24. Solve the initial value problem

\[ y' = (2 - e^{-y})/(3 + 2y), \quad y(0) = 0 \]

and determine where the solution attains its maximum value.

25. Solve the initial value problem

\[ y' = 2 \cos 2x/(3 + 2y), \quad y(0) = -1 \]

and determine where the solution attains its maximum value.

26. Solve the initial value problem

\[ y' = 2(1 + x)(1 + y^2), \quad y(0) = 0 \]

and determine where the solution attains its minimum value.
2. Separable Equations

27. Consider the initial value problem
\[ y' = ty(4 - y)/3, \quad y(0) = y_0. \]
(a) Determine how the behavior of the solution as \( t \) increases depends on the initial value \( y_0 \).
(b) Suppose that \( y_0 = 0.5 \). Find the time \( T \) at which the solution first reaches the value 3.98.

28. Consider the initial value problem
\[ y' = ty(4 - y)/(1 + t), \quad y(0) = y_0 > 0. \]
(a) Determine how the solution behaves as \( t \to +\infty \).
(b) If \( y_0 = 2 \), find the time \( T \) at which the solution first reaches the value 3.99.
(c) Find the range of initial values for which the solution lies in the interval 3.99 < \( y \) < 4.01 by the time \( t = 2 \).

29. Solve the equation
\[ \frac{dy}{dx} = \frac{ay + b}{cy + d}, \]
where \( a, b, c, \) and \( d \) are constants.

Homogeneous Equations. If the right side of the equation \( dy/dx = f(x, y) \) can be expressed as a function of the ratio \( y/x \) only, then the equation is said to be homogeneous.\(^1\) Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 30 illustrates how to solve first order homogeneous equations.

30. Consider the equation
\[ \frac{dy}{dx} = \frac{y - 4x}{x - y}. \]
(a) Show that Eq. (i) can be rewritten as
\[ \frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}. \]
thus Eq. (i) is homogeneous.
(b) Introduce a new dependent variable \( v \) so that \( v = y/x \), or \( y = xv(x) \). Express \( dy/dx \) in terms of \( x, v, \) and \( dv/dx \).
(c) Replace \( y \) and \( dy/dx \) in Eq. (ii) by the expressions from part (b) that involve \( v \) and \( dv/dx \). Show that the resulting differential equation is
\[ v + x \frac{dv}{dx} = \frac{v - 4}{1 - v}, \]
or
\[ \frac{dv}{dx} = \frac{v^2 - 4}{1 - v}. \]

Observe that Eq. (iii) is separable.

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\(^1\)The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.
Chapter 2. First Order Differential Equations

(d) Solve Eq. (iii), obtaining \( v \) implicitly in terms of \( x \).
(e) Find the solution of Eq. (i) by replacing \( v \) by \( y/x \) in the solution in part (d).
(f) Draw a direction field and some integral curves for Eq. (i). Recall that the right side of Eq. (i) actually depends only on the ratio \( y/x \). This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution \( y = xu(x) \) transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing \( u \) by \( y/x \) gives the solution to the original equation. In each of Problems 31 through 38:
(a) Show that the given equation is homogeneous.
(b) Solve the differential equation.
(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

31. \( \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \)
32. \( \frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} \)
33. \( \frac{dy}{dx} = \frac{4y - 3x}{2x - y} \)
34. \( \frac{dy}{dx} = -\frac{4x + 3y}{2x + y} \)
35. \( \frac{dy}{dx} = \frac{x + 3y}{x - y} \)
36. \( \frac{dy}{dx} = -\frac{x^2 + 3xy + y^2}{x^2} \)
37. \( \frac{dy}{dx} = \frac{x^2 - 3y^2}{2xy} \)
38. \( \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \)

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.
and Eq. (26) is replaced by
\[ \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}. \]  
(28)

Equation (28) is separable but not linear, so by separating the variables and integrating, we obtain
\[ \frac{v^2}{2} = \frac{gR^2}{R+x} + c. \]  
(29)

Since \( x = 0 \) when \( t = 0 \), the initial condition (27) at \( t = 0 \) can be replaced by the condition that \( v = v_0 \) when \( x = 0 \). Hence \( c = \left(\frac{v_0^2}{2}\right) - gR \) and
\[ v = \pm \sqrt{\frac{v_0^2}{2} - 2gR + \frac{2gR^2}{R+x}}. \]  
(30)

Note that Eq. (30) gives the velocity as a function of altitude rather than as a function of time. The plus sign must be chosen if the body is rising, and the minus sign if it is falling back to earth.

To determine the maximum altitude \( \xi \) that the body reaches, we set \( v = 0 \) and \( x = \xi \) in Eq. (30) and then solve for \( \xi \), obtaining
\[ \xi = \frac{v_0^2R}{2gR - v_0^2}. \]  
(31)

Solving Eq. (31) for \( v_0 \), we find the initial velocity required to lift the body to the altitude \( \xi \), namely,
\[ v_0 = \sqrt{\frac{2gR}{R+\xi}}. \]  
(32)

The escape velocity \( v_e \) is then found by letting \( \xi \to \infty \). Consequently,
\[ v_e = \sqrt{2gR}. \]  
(33)

The numerical value of \( v_e \) is approximately 6.9 m/s, or 11.1 km/s.

The preceding calculation of the escape velocity neglects the effect of air resistance, so the actual escape velocity (including the effect of air resistance) is somewhat higher. On the other hand, the effective escape velocity can be significantly reduced if the body is transported a considerable distance above sea level before being launched. Both gravitational and frictional forces are thereby reduced; air resistance, in particular, diminishes quite rapidly with increasing altitude. You should keep in mind also that it may well be impractical to impart too large an initial velocity instantaneously; space vehicles, for instance, receive their initial acceleration during a period of a few minutes.

### PROBLEMS

1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of \( \gamma \) g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of \( \gamma \) for the amount of salt in the tank at any time \( t \). Also find the limiting amount of salt in the tank as \( t \to \infty \).
3. A tank originally contains 100 gal of fresh water. Then water containing \( \frac{1}{2} \) lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

5. A tank contains 100 gal of water and 50 oz of salt. Water containing a salt concentration of \( \frac{1}{2} (1 + \frac{1}{2} \sin t) \text{ oz/gal} \) flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.
   (a) Find the amount of salt in the tank at any time.
   (b) Plot the solution for a time period long enough so that you see the ultimate behavior of the graph.
   (c) The long-time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

6. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let \( h(t) \) be the height of the liquid surface above the outlet at time \( t \). Torricelli's \(^2\) principle states that the outflow velocity \( v \) at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height \( h \).
   (a) Show that \( v = \sqrt{2gh} \), where \( g \) is the acceleration due to gravity.
   (b) By equating the rate of outflow to the rate of change of liquid in the tank, show that \( h(t) \) satisfies the equation
      \[
      A(h) \frac{dh}{dt} = -a \sqrt{2gh},
      \]
      where \( A(h) \) is the area of the cross section of the tank at height \( h \) and \( a \) is the area of the outlet. The constant \( a \) is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than \( a \). The value of \( a \) for water is about 0.6.
   (c) Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

7. Suppose that a sum \( S_0 \) is invested at an annual rate of return \( r \) compounded continuously.
   (a) Find the time \( T \) required for the original sum to double in value as a function of \( r \).
   (b) Determine \( T \) if \( r = 7\% \).
   (c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

---

\(^2\) Evangelista Torricelli (1608–1647), successor to Galileo as court mathematician in Florence, published this result in 1644. He is also known for constructing the first mercury barometer and for making important contributions to geometry.
8. A young person with no initial capital invests $k$ dollars per year at an annual rate of return $r$. Assume that investments are made continuously and that the return is compounded continuously.

(a) Determine the sum $S(t)$ accumulated at any time $t$.
(b) If $r = 7.5\%$, determine $k$ so that $1$ million will be available for retirement in 40 years.
(c) If $k \neq 2000\$/year, determine the return rate $r$ that must be obtained to have $1$ million available in 40 years.

9. A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of $10\%$. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate $k$, determine the payment rate $k$ that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

10. A home buyer can afford to spend no more than $800/month on mortgage payments. Suppose that the interest rate is $9\%$ and that the term of the mortgage is 20 years. Assume that interest is compounded continuously and that payments are also made continuously.

(a) Determine the maximum amount that this buyer can afford to borrow.
(b) Determine the total interest paid during the term of the mortgage.

11. A recent college graduate borrows $100,000 at an interest rate of $9\%$ to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + r/120)$, where $r$ is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?
(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

12. An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby.\(^3\) This is a means of determining the age of certain wood and plant remains, hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years\(^4\)), measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the proportion of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time $t$ and $Q_0$ is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

(a) Assuming that $Q$ satisfies the differential equation $Q' = -rQ$, determine the decay constant $r$ for carbon-14.
(b) Find an expression for $Q(t)$ at any time $t$, if $Q(0) = Q_0$.

---

\(^3\) Willard F. Libby (1908–1980) was born in rural Colorado and received his education at the University of California at Berkeley. He developed the method of radiocarbon dating beginning in 1947 while he was at the University of Chicago. For this work, he was awarded the Nobel Prize in chemistry in 1960.

Chapter 2. First Order Differential Equations

(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

13. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

Suppose that a certain population has a growth rate that varies with time and that this population satisfies the differential equation

\[ \frac{dy}{dt} = (0.5 + \sin t)y/5. \]

(a) If \( y(0) = 1 \), find (or estimate) the time \( t \) at which the population has doubled. Choose other initial conditions and determine whether the doubling time \( t \) depends on the initial population.

(b) Suppose that the growth rate is replaced by its average value \( 1/10 \). Determine the doubling time \( t \) in this case.

(c) Suppose that the term \( \sin t \) in the differential equation is replaced by \( \sin 2n t \); that is, the variation in the growth rate has a substantially higher frequency. What effect does this have on the doubling time \( t \)?

(d) Plot the solutions obtained in parts (a), (b), and (c) on a single set of axes.

15. Suppose that a certain population satisfies the initial value problem

\[ \frac{dy}{dt} = r(t)y - k, \quad y(0) = y_0, \]

where the growth rate \( r(t) \) is given by \( r(t) = (1 + \sin t)/5 \), and \( k \) represents the rate of predation.

(a) Suppose that \( k = 1/5 \). Plot \( y \) versus \( t \) for several values of \( y_0 \) between 1/2 and 1.

(b) Estimate the critical initial population \( y_0 \) below which the population will become extinct.

(c) Choose other values of \( k \) and find the corresponding \( y_0 \) for each one.

(d) Use the data you have found in parts (b) and (c) to plot \( y_0 \) versus \( k \).

16. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

17. Heat transfer from a body to its surroundings by radiation, based on the Stefan–Boltzmann law, is described by the differential equation

\[ \frac{du}{dt} = -\alpha (u^4 - T^4), \]

where \( u(t) \) is the absolute temperature of the body at time \( t \), \( T \) is the absolute temperature of the surroundings, and \( \alpha \) is a constant depending on the physical parameters of the body.

\textsuperscript{5}Josef Stefan (1835–1893), professor of physics at Vienna, stated the radiation law on empirical grounds in 1879. His student Ludwig Boltzmann (1844–1906) derived it theoretically from the principles of thermodynamics in 1884. Boltzmann is best known for his pioneering work in statistical mechanics.
However, if $u$ is much larger than $T$, then solutions of Eq. (i) are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4. \quad (ii)$$

Suppose that a body with initial temperature $2000^\circ\text{K}$ is surrounded by a medium with temperature $300^\circ\text{K}$ and that $\alpha = 2.0 \times 10^{-12} \text{ K}^{-3/8}$.

(a) Determine the temperature of the body at any time by solving Eq. (ii).

(b) Plot the graph of $u$ versus $t$.

(c) Find the time $\tau$ at which $u(\tau) = 600$, that is, twice the ambient temperature. Up to this time the error in using Eq. (ii) to approximate the solutions of Eq. (i) is no more than 1%.

18. Consider an insulated box (a building, perhaps) with internal temperature $u(t)$. According to Newton's law of cooling, $u$ satisfies the differential equation

$$\frac{du}{dt} = -k(u - T(t)), \quad (i)$$

where $T(t)$ is the ambient (external) temperature. Suppose that $T(t)$ varies sinusoidally; for example, assume that $T(t) = T_0 + T_1 \cos \omega t$.

(a) Solve Eq. (i) and express $u(t)$ in terms of $i, k, T_0, T_1, \omega$. Observe that part of your solution approaches zero as $t$ becomes large; this is called the transient part. The remainder of the solution is called the steady state; denote it by $S(t)$.

(b) Suppose that $t$ is measured in hours and that $\omega = \pi/12$, corresponding to a period of 24 h for $T(t)$. Further, let $T_0 = 60^\circ\text{F}, T_1 = 15^\circ\text{F}$, and $k = 0.2/h$. Draw graphs of $S(t)$ and $T(t)$ versus $t$ on the same axes. From your graph estimate the amplitude $R$ of the oscillatory part of $S(t)$. Also estimate the time lag $\tau$ between corresponding maxima of $T(t)$ and $S(t)$.

(c) Let $k, T_0, T_1,$ and $\omega$ now be unspecified. Write the oscillatory part of $S(t)$ in the form $R \cos(\omega(t - \tau))$. Use trigonometric identities to find expressions for $R$ and $\tau$. Let $T_1$ and $\omega$ have the values given in part (b), and plot graphs of $R$ and $\tau$ versus $k$.

19. Consider a lake of constant volume $V$ containing at time $t = 0$ an amount $Q(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = Q(t)/V$. Assume that water containing a concentration $k$ of pollutant enters the lake at a rate $r$, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate $P$. Note that the given assumptions neglect a number of factors that may, in some cases, be important—for example, the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are not deposited evenly throughout the lake but (usually) at isolated points around its periphery. The results below must be interpreted in the light of the neglect of such factors as these.

(a) If at time $t = 0$ the concentration of pollutant is $c_0$, find an expression for the concentration $c(t)$ at any time. What is the limiting concentration as $t \to \infty$?

(b) If the addition of pollutants to the lake is terminated $(k = 0$ and $P = 0$ for $t > 0$), determine the time interval $T$ that must elapse before the concentration of pollutants is reduced to 50% of its original value; to 10% of its original value.

(c) Table 2.3.2 contains data$^6$ for several of the Great Lakes. Using these data, determine
from part (b) the time \( T \) necessary to reduce the contamination of each of these lakes to 10% of the original value.

**Table 2.3.2 Volume and Flow Data for the Great Lakes**

<table>
<thead>
<tr>
<th>Lake</th>
<th>( V ) (km(^3) x 10(^5))</th>
<th>( r ) (km(^3)/year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superior</td>
<td>12.2</td>
<td>65.2</td>
</tr>
<tr>
<td>Michigan</td>
<td>4.9</td>
<td>158</td>
</tr>
<tr>
<td>Erie</td>
<td>0.46</td>
<td>175</td>
</tr>
<tr>
<td>Ontario</td>
<td>1.6</td>
<td>209</td>
</tr>
</tbody>
</table>

\( \star \) 20. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

(a) Find the maximum height above the ground that the ball reaches.

(b) Assuming that the ball misses the building on the way down, find the time that it hits the ground.

(c) Plot the graphs of velocity and position versus time.

\( \star \) 21. Assume that the conditions are as in Problem 20 except that there is a force due to air resistance of \(|v|/30\), where the velocity \( v \) is measured in m/s.

(a) Find the maximum height above the ground that the ball reaches.

(b) Find the time that the ball hits the ground.

(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problem 20.

\( \star \) 22. Assume that the conditions are as in Problem 20 except that there is a force due to air resistance of \( v^2/1325 \), where the velocity \( v \) is measured in m/s.

(a) Find the maximum height above the ground that the ball reaches.

(b) Find the time that the ball hits the ground.

(c) Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 20 and 21.

\( \star \) 23. A sky diver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is 0.75|\( v \)| when the parachute is closed and 12|\( v \)| when the parachute is open, where the velocity \( v \) is measured in ft/s.

(a) Find the speed of the sky diver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

(c) What is the limiting velocity \( v_L \) after the parachute opens?

(d) Determine how long the sky diver is in the air after the parachute opens.

(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

24. A rocket sled having an initial speed of 150 mi/h is slowed by a channel of water. Assume that, during the braking process, the acceleration \( a \) is given by \( a(v) = -\mu v^2 \), where \( v \) is the velocity and \( \mu \) is a constant.

(a) As in Example 4 in the text, use the relation \( dv/dt = v(du/dx) \) to write the equation of motion in terms of \( v \) and \( x \).

(b) If it requires a distance of 2000 ft to slow the sled to 15 mi/h, determine the value of \( \mu \).

(c) Find the time \( t \) required to slow the sled to 15 mi/h.
25. A body of constant mass \( m \) is projected vertically upward with an initial velocity \( v_0 \) in a medium offering a resistance \( kv \), where \( k \) is a constant. Neglect changes in the gravitational force.

(a) Find the maximum height \( x_m \) attained by the body and the time \( t_m \) at which this maximum height is reached.

(b) Show that if \( kv_0/\frac{mg}{2} < 1 \), then \( t_m \) and \( x_m \) can be expressed as

\[
\begin{align*}
  t_m &= \frac{v_0}{\frac{mg}{2}} \left[ 1 - \frac{1}{2} \frac{kv_0}{mg} + \frac{1}{3} \left( \frac{kv_0}{mg} \right)^2 - \ldots \right], \\
  x_m &= \frac{v_0^2}{2g} \left[ 1 - \frac{2}{3} \frac{kv_0}{mg} + \frac{1}{2} \left( \frac{kv_0}{mg} \right)^2 - \ldots \right].
\end{align*}
\]

(c) Show that the quantity \( kv_0/\frac{mg}{2} \) is dimensionless.

26. A body of mass \( m \) is projected vertically upward with an initial velocity \( v_0 \) in a medium offering a resistance \( kv \), where \( k \) is a constant. Assume that the gravitational attraction of the earth is constant.

(a) Find the velocity \( v(t) \) of the body at any time.

(b) Use the result of part (a) to calculate the limit of \( v(t) \) as \( k \to 0 \), that is, as the resistance approaches zero. Does this result agree with the velocity of a mass \( m \) projected upward with an initial velocity \( v_0 \) in a vacuum?

(c) Use the result of part (a) to calculate the limit of \( v(t) \) as \( m \to 0 \), that is, as the mass approaches zero.

27. A body falling in a relatively dense fluid, oil for example, is acted on by three forces (see Figure 2.3.5): a resistive force \( R \), a buoyant force \( B \), and its weight \( w \) due to gravity. The buoyant force is equal to the weight of the fluid displaced by the object. For a slowly moving spherical body of radius \( a \), the resistive force is given by Stokes's law, \( R = 6\pi \mu a|v| \), where \( v \) is the velocity of the body, and \( \mu \) is the coefficient of viscosity of the surrounding fluid.\(^7\)

![Figure 2.3.5 A body falling in a dense fluid.](image)

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\(^7\)George Gabriel Stokes (1819–1903), professor at Cambridge, was one of the foremost applied mathematicians of the nineteenth century. The basic equations of fluid mechanics (the Navier–Stokes equations) are named partly in his honor, and one of the fundamental theorems of vector calculus bears his name. He was also one of the pioneers in the use of divergent (asymptotic) series, a subject of great interest and importance today.
(a) Find the limiting velocity of a solid sphere of radius \( a \) and density \( \rho \) falling freely in a medium of density \( \rho' \) and coefficient of viscosity \( \mu \).

(b) In 1910 R. A. Millikan\(^8\) studied the motion of tiny droplets of oil falling in an electric field. A field of strength \( E \) exerts a force \( Ee \) on a droplet with charge \( e \). Assume that \( E \) has been adjusted so the droplet is held stationary (\( v = 0 \)) and that \( w \) and \( B \) are as given above. Find an expression for \( e \). Millikan repeated this experiment many times, and from the data that he gathered he was able to deduce the charge on an electron.

28. A mass of 0.25 kg is dropped from rest in a medium offering a resistance of 0.2|\( v \)|, where \( v \) is measured in m/s.
   (a) If the mass is dropped from a height of 30 m, find its velocity when it hits the ground.
   (b) If the mass is to attain a velocity of no more than 10 m/s, find the maximum height from which it can be dropped.
   (c) Suppose that the resistive force is \( k|v| \), where \( v \) is measured in m/s and \( k \) is a constant. If the mass is dropped from a height of 30 m and must hit the ground with a velocity of no more than 10 m/s, determine the coefficient of resistance \( k \) that is required.

29. Suppose that a rocket is launched straight up from the surface of the earth with initial velocity \( v_0 = \sqrt{2gR} \), where \( R \) is the radius of the earth. Neglect air resistance.
   (a) Find an expression for the velocity \( v \) in terms of the distance \( x \) from the surface of the earth.
   (b) Find the time required for the rocket to go 240,000 mi (the approximate distance from the earth to the moon). Assume that \( R = 4000 \) mi.

2. Let \( v(t) \) and \( w(t) \), respectively, be the horizontal and vertical components of the velocity of a batted (or thrown) baseball. In the absence of air resistance, \( v \) and \( w \) satisfy the equations

\[
\frac{dv}{dt} = 0, \quad \frac{dw}{dt} = -g.
\]

(a) Show that

\[
v = u \cos A, \quad w = -gt + u \sin A,
\]

where \( u \) is the initial speed of the ball and \( A \) is its initial angle of elevation.

(b) Let \( x(t) \) and \( y(t) \), respectively, be the horizontal and vertical coordinates of the ball at time \( t \). If \( x(0) = 0 \) and \( y(0) = h \), find \( x(t) \) and \( y(t) \) at any time \( t \).

(c) Let \( g = 32 \) ft/s\(^2\), \( u = 125 \) ft/s, and \( h = 3 \) ft. Plot the trajectory of the ball for several values of the angle \( A \); that is, plot \( x(t) \) and \( y(t) \) parametrically.

(d) Suppose the outfield wall is at a distance \( L \) and has height \( H \). Find a relation between \( u \) and \( A \) that must be satisfied if the ball is to clear the wall.

(e) Suppose that \( L = 350 \) ft and \( H = 10 \) ft. Using the relation in part (d), find (or estimate from a plot) the range of values of \( A \) that correspond to an initial velocity of \( u = 110 \) ft/s.

(f) For \( L = 350 \) and \( H = 10 \), find the minimum initial velocity \( u \) and the corresponding optimal angle \( A \) for which the ball will clear the wall.

21. A more realistic model (than that in Problem 30) of a baseball in flight includes the effect of air resistance. In this case the equations of motion are

\[
\frac{dv}{dt} = -rv, \quad \frac{dw}{dt} = -g - rw,
\]

---

\(^8\)Robert A. Millikan (1868–1953) was educated at Oberlin College and Columbia University. Later he was a professor at the University of Chicago and California Institute of Technology. His determination of the charge on an electron was published in 1910. For this work, and for other studies of the photoelectric effect, he was awarded the Nobel Prize in 1923.
2.3 Modeling with First Order Equations

where \( r \) is the coefficient of resistance.

(a) Determine \( v(t) \) and \( w(t) \) in terms of initial speed \( u \) and initial angle of elevation \( A \).
(b) Find \( x(t) \) and \( y(t) \) if \( x(0) = 0 \) and \( y(0) = h \).
(c) Plot the trajectory of the ball for \( r = 1/5 \), \( u = 125 \), \( h = 3 \), and for several values of \( A \). How do the trajectories differ from those in Problem 31 with \( r = 0 \)?
(d) Assuming that \( r = 1/5 \) and \( h = 3 \), find the minimum initial velocity \( u \) and the optimal angle \( A \) for which the ball will clear a wall that is 350 ft distant and 10 ft high. Compare this result with that in Problem 30(f).

32. Brachistochrone Problem. One of the famous problems in the history of mathematics is the brachistochrone\(^9\) problem: to find the curve along which a particle will slide without friction in the minimum time from one given point \( P \) to another \( Q \), the second point being lower than the first but not directly beneath it (see Figure 2.3.6). This problem was posed by Johann Bernoulli in 1696 as a challenge problem to the mathematicians of his day. Correct solutions were found by Johann Bernoulli and his brother Jakob Bernoulli and by Isaac Newton, Gottfried Leibniz, and the Marquis de L'Hospital. The brachistochrone problem is important in the development of mathematics as one of the forerunners of the calculus of variations.

In solving this problem, it is convenient to take the origin as the upper point \( P \) and to orient the axes as shown in Figure 2.3.6. The lower point \( Q \) has coordinates \((x_0, y_0)\). It is then possible to show that the curve of minimum time is given by a function \( y = \phi(x) \) that satisfies the differential equation

\[
(1 + y'^2)y = k^2,
\]

where \( k^2 \) is a certain positive constant to be determined later.

Figure 2.3.6 The brachistochrone.

(a) Solve Eq. (i) for \( y' \). Why is it necessary to choose the positive square root?
(b) Introduce the new variable \( t \) by the relation

\[
y = k^2 \sin^2 t.
\]

Show that the equation found in part (a) then takes the form

\[
2k^2 \sin^3 t \, dt = dx.
\]

\(^9\)The word "brachistochrone" comes from the Greek words \textit{brachistos}, meaning shortest, and \textit{chronos}, meaning time.
fying regions of the $ty$-plane where solutions exhibit interesting features that merit more detailed analytical or numerical investigation. Graphical methods for first order equations are discussed further in Section 2.5. An introduction to numerical methods for first order equations is given in Section 2.7, and a systematic discussion of numerical methods appears in Chapter 8. However, it is not necessary to study the numerical algorithms themselves in order to use effectively one of the many software packages that generate and plot numerical approximations to solutions of initial value problems.

**Summary.** The linear equation $y' + p(t)y = g(t)$ has several nice properties that can be summarized in the following statements:

1. Assuming that the coefficients are continuous, there is a general solution, containing an arbitrary constant, that includes all solutions of the differential equation. A particular solution that satisfies a given initial condition can be picked out by choosing the proper value for the arbitrary constant.
2. There is an expression for the solution, namely, Eq. (7) or Eq. (8). Moreover, although it involves two integrations, the expression is an explicit one for the solution $y = \phi(t)$ rather than an equation that defines $\phi$ implicitly.
3. The possible points of discontinuity or singularities of the solution can be identified (without solving the problem) merely by finding the points of discontinuity of the coefficients. Thus, if the coefficients are continuous for all $t$, then the solution also exists and is differentiable for all $t$.

None of these statements is true, in general, of nonlinear equations. Although a nonlinear equation may well have a solution involving an arbitrary constant, there may also be other solutions. There is no general formula for solutions of nonlinear equations. If you are able to integrate a nonlinear equation, you are likely to obtain an equation defining solutions implicitly rather than explicitly. Finally, the singularities of solutions of nonlinear equations can usually be found only by solving the equation and examining the solution. It is likely that the singularities will depend on the initial condition as well as the differential equation.

**PROBLEMS**

In each of Problems 1 through 6 determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t - 3)y' + (\ln t)y = 2t, \quad y(1) = 2$
2. $t(t - 4)y' + y = 0, \quad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$
4. $(4 - t^2)y' + 2ty = 3t^2, \quad y(-3) = 1$
5. $(4 - t^2)y' + 2ty = 3t^2, \quad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \quad y(2) = 3$

In each of Problems 7 through 12 state where in the $ty$-plane the hypotheses of Theorem 2.4.2 are satisfied.

7. $y' = \frac{t - y}{2t + 5y}$
8. $y' = (1 - t^2 - y^2)^{1/2}$
9. $y' = \frac{\ln |ty|}{1 - t^2 + y^2}$
10. $y' = (t^2 + y^2)^{1/2}$
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11. \[ \frac{dy}{dt} = \frac{1 + r^2}{3y - y^3} \]

12. \[ \frac{dy}{dt} = \frac{\cot t \cdot y}{1 + y} \]

In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value \( y_0 \).

13. \( y' = -4t/y \), \( y(0) = y_0 \)
14. \( y' = 2t^2 \), \( y(0) = y_0 \)
15. \( y' + y^3 = 0 \), \( y(0) = y_0 \)
16. \( y' = t^2/y(1 + r^2) \), \( y(0) = y_0 \)

In each of Problems 17 through 20 draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as \( t \) increases and how their behavior depends on the initial value \( y_0 \) when \( t = 0 \).

17. \( y' = ty(3 - y) \)
18. \( y' = y(3 - ty) \)
19. \( y' = -y(3 - ty) \)
20. \( y' = t - 1 - y^2 \)

21. Consider the initial value problem \( y' = y^{1/3}, y(0) = 0 \) from Example 3 in the text.
   (a) Is there a solution that passes through the point \((1,1)\)? If so, find it.
   (b) Is there a solution that passes through the point \((2,1)\)? If so, find it.
   (c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions have at \( t = 2 \).

22. (a) Verify that both \( y_1(t) = 1 - t \) and \( y_2(t) = -t^2/4 \) are solutions of the initial value problem

\[ y' = \frac{t + (t^2 + 4y)^{1/2}}{2}, \quad y(2) = -1. \]

Where are these solutions valid?
(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
(c) Show that \( y = ct + c^2 \), where \( c \) is an arbitrary constant, satisfies the differential equation in part (a) for \( t \geq -2c \). If \( c = -1 \), the initial condition is also satisfied, and the solution \( y = y_1(t) \) is obtained. Show that there is no choice of \( c \) that gives the second solution \( y = y_2(t) \).

23. (a) Show that \( \phi(t) = e^{ct} \) is a solution of \( y' - 2y = 0 \) and that \( y = c\phi(t) \) is also a solution of this equation for any value of the constant \( c \).
(b) Show that \( \phi(t) = 1/t \) is a solution of \( y' + y^2 = 0 \) for \( t > 0 \) but that \( y = c\phi(t) \) is not a solution of this equation unless \( c = 0 \) or \( c = 1 \). Note that the equation of part (b) is nonlinear, while that of part (a) is linear.

24. Show that if \( y = \phi(t) \) is a solution of \( y' + p(t)y = 0 \), then \( y = c\phi(t) \) is also a solution for any value of the constant \( c \).

25. Let \( y = y_1(t) \) be a solution of \( y' + p(t)y = 0 \), and let \( y = y_2(t) \) be a solution of \( y' + p(t)y = g(t) \),

\[ y' + p(t)y = g(t). \]

Show that \( y = y_1(t) + y_2(t) \) is also a solution of Eq. (ii).

26. (a) Show that the solution (7) of the general linear equation (1) can be written in the form

\[ y = cy_1(t) + y_2(t), \]

where \( c \) is an arbitrary constant. Identify the functions \( y_1 \) and \( y_2 \).
(b) Show that $y_1$ is a solution of the differential equation
\[ y' + p(t)y = 0, \]
(ii)
corresponding to $g(t) = 0$.

(c) Show that $y_2$ is a solution of the full linear equation (1). We see later (for example, in Section 3.5) that solutions of higher order linear equations have a pattern similar to Eq. (i).

**Bernoulli Equations.** Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form
\[ y' + p(t)y = q(t)y^n, \]
and is called a Bernoulli equation after Jakob Bernoulli. Problems 27 through 31 deal with equations of this type.

27. (a) Solve Bernoulli's equation when $n = 0$; when $n = 1$.
(b) Show that if $n \neq 0, 1$, then the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation. This method of solution was found by Leibniz in 1696.

In each of Problems 28 through 31 the given equation is a Bernoulli equation. In each case solve it by using the substitution mentioned in Problem 27(b).

28. $t^2y' + 2ty - y^3 = 0, \quad t > 0$

29. $y' = ry - ky^2, \quad r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.

30. $y' = ey - bx^2, \quad e > 0$ and $b > 0$. This equation occurs in the study of the stability of fluid flow.

31. $dy/dt = (t \cos t + T)y - y^3$, where $T$ and $T$ are constants. This equation also occurs in the study of the stability of fluid flow.

**Discontinuous Coefficients.** Linear differential equations sometimes occur in which one or both of the functions $p$ and $g$ have jump discontinuities. If $t_0$ is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that $y$ is continuous at $t_0$; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Note in each case that it is impossible also to make $y'$ continuous at $t_0$.

32. Solve the initial value problem
\[ y' + 2y = g(t), \quad y(0) = 0, \]
where
\[ g(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & t > 1. \end{cases} \]

33. Solve the initial value problem
\[ y' + p(t)y = 0, \quad y(0) = 1, \]
where
\[ p(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ 1, & t > 1. \end{cases} \]
Thus, if \((\mu M)_y\) is to equal \((\mu N)_x\), it is necessary that

\[
\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu. \tag{27}
\]

If \((M_y - N_x)/N\) is a function of \(x\) only, then there is an integrating factor \(\mu\) that also depends only on \(x\); further, \(\mu(x)\) can be found by solving Eq. (27), which is both linear and separable.

A similar procedure can be used to determine a condition under which Eq. (23) has an integrating factor depending only on \(y\); see Problem 23.

**Example 4**

Find an integrating factor for the equation

\[
(3xy + y^2) + (x^2 + xy)y' = 0 \tag{19}
\]

and then solve the equation.

In Example 3 we showed that this equation is not exact. Let us determine whether it has an integrating factor that depends on \(x\) only. On computing the quantity \((M_y - N_x)/N\), we find that

\[
\frac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}. \tag{28}
\]

Thus there is an integrating factor \(\mu\) that is a function of \(x\) only, and it satisfies the differential equation

\[
\frac{d\mu}{dx} = \frac{\mu}{x}. \tag{29}
\]

Hence

\[
\mu(x) = x. \tag{30}
\]

Multiplying Eq. (19) by this integrating factor, we obtain

\[
(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{31}
\]

The latter equation is exact, and its solutions are given implicitly by

\[
x^3y + \frac{1}{2}x^2y^2 = c. \tag{32}
\]

Solutions may also be found in explicit form since Eq. (32) is quadratic in \(y\).

You may also verify that a second integrating factor for Eq. (19) is

\[
\mu(x,y) = \frac{1}{xy(2x + y)},
\]

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

**Problems**

Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

1. \((2x + 3) + (2y - 2)y' = 0\)
2. \((2x + 4y) + (2x - 2y)y' = 0\)
3. \((3x^2 - 2xy + 2) dx + (6y^2 - x^2 + 3) dy = 0\)
4. \((2xy^2 + 2y) + (2x^2y + 2x)y' = 0\)
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5. \( \frac{dy}{dx} = \frac{-ax + by}{bx + cy} \)
6. \( \frac{dy}{dx} = \frac{ax - by}{bx - cy} \)
7. \( (e^y \sin y - 2y \sin x) \, dx + (e^y \cos y + 2 \cos x) \, dy = 0 \)
8. \( (e^y \sin y + 3y) \, dx + (3x - e^y \sin y) \, dy = 0 \)
9. \( (ye^y \cos 2x - 2e^y \sin 2x + 2x) \, dx + (xe^y \cos 2x - 3) \, dy = 0 \)
10. \( (y/x + 6x) \, dx + (\ln x - 2) \, dy = 0, \quad x > 0 \)
11. \( (x \ln y + xy) \, dx + (y \ln x + xy) \, dy = 0, \quad x > 0, \quad y > 0 \)
12. \( \frac{x \, dx}{(x^2 + y^2)^{3/2}} + \frac{y \, dy}{(x^2 + y^2)^{3/2}} = 0 \)

In each of Problems 13 and 14 solve the given initial value problem and determine at least approximately where the solution is valid.

13. \( (2x - y) \, dx + (2y - x) \, dy = 0, \quad y(1) = 3 \)
14. \( (9x^2 + y - 1) \, dx - (4y - x) \, dy = 0, \quad y(1) = 0 \)

In each of Problems 15 and 16 find the value of \( b \) for which the given equation is exact, and then solve it using that value of \( b \).

15. \( (y^2 + bx^2y) \, dx + (x + y)x^2 \, dy = 0 \)
16. \( (ye^{2y} + x) \, dx + bxe^{2y} \, dy = 0 \)

17. Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle \( R \) and is therefore exact. Show that a possible function \( \psi(x, y) \) is

\[ \psi(x, y) = \int_{x_0}^{x} M(x, y_0) \, dx + \int_{y_0}^{y} N(x, t) \, dt, \]

where \((x_0, y_0)\) is a point in \( R \).

18. Show that any separable equation

\[ M(x) + N(y)y' = 0 \]

is also exact.

In each of Problems 19 through 22 show that the given equation is not exact but becomes exact when multiplied by the given integrating factor. Then solve the equation.

19. \( x^2y^3 + x(1 + y^2)y' = 0, \quad \mu(x, y) = 1/xy^3 \)
20. \( \left( \frac{\sin y}{y} - 2e^{-2x} \sin x \right) \, dx + \left( \frac{\cos y + 2e^{-x} \cos x}{y} \right) \, dy = 0, \quad \mu(x, y) = ye^x \)
21. \( y \, dx + (2x - ye^y) \, dy = 0, \quad \mu(x, y) = y \)
22. \( (x + 2) \sin y \, dx + x \cos y \, dy = 0, \quad \mu(x, y) = xe^y \)

23. Show that if \((N_y - M_x)/M = Q\), where \( Q \) is a function of \( y \) only, then the differential equation

\[ M + Ny' = 0 \]

has an integrating factor of the form

\[ \mu(y) = \exp \int Q(y) \, dy. \]

24. Show that if \((N_y - M_x)/(xM - yN) = R\), where \( R \) depends on the quantity \( xy \) only, then the differential equation

\[ M + Ny' = 0 \]

has an integrating factor of the form \( \mu(xy) \). Find a general formula for this integrating factor.
2.7 Numerical Approximations: Euler’s Method

In each of Problems 25 through 31 find an integrating factor and solve the given equation.

25. \((3x^2 + 2xy + y^2) \, dx + (x^2 + y^2) \, dy = 0\)  \quad 26. \(y' = e^{2x} - y - 1\)

27. \(dx + (x/y - \sin y) \, dy = 0\)  \quad 28. \(y \, dx + (2xy - e^{-2y}) \, dy = 0\)

29. \(e^x \, dx + (e^x \cos y + 2y \sec y) \, dy = 0\)

30. \([4(x^2/y^2) + (3/y)] \, dx + [3(x/y^2) + 4y] \, dy = 0\)

31. \(\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3y\right) \, \frac{dy}{dx} = 0\)

*Hint: See Problem 24.*

32. Solve the differential equation

\[(3xy + y^2) + (x^2 + xy)y' = 0\]

using the integrating factor \(\mu(x, y) = \left[xy(2x + y)^{-1}\right]\). Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

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2.7 Numerical Approximations: Euler’s Method

Recall two important facts about the first order initial value problem

\[
\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{1}
\]

First, if \(f\) and \(\partial f/\partial y\) are continuous, then the initial value problem (1) has a unique solution \(y = \phi(t)\) in some interval surrounding the initial point \(t = t_0\). Second, it is usually not possible to find the solution \(\phi\) by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to the latter statement: differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means, such as those considered in the first part of this chapter.

Therefore it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated differential equations. However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.

For example, Figure 2.7.1 shows a direction field for the differential equation

\[
\frac{dy}{dt} = 3 - 2t - \frac{1}{2}y. \tag{2}
\]

From the direction field you can visualize the behavior of solutions on the rectangle shown in the figure. A solution starting at a point on the \(y\)-axis initially increases with \(t\), but it soon reaches a maximum value and then begins to decrease as \(t\) increases further.
Chapter 3. Second Order Linear Equations

PROBLEMS

In each of Problems 1 through 8 find the general solution of the given differential equation.

1. \( y'' + 2y' - 3y = 0 \)
2. \( y'' + 3y' + 2y = 0 \)
3. \( 6y'' - y' - y = 0 \)
4. \( 2y'' - 3y' + y = 0 \)
5. \( y'' + 5y' = 0 \)
6. \( 4y'' - 9y = 0 \)
7. \( y'' - 9y' + 9y = 0 \)
8. \( y'' - 2y' - 2y = 0 \)

In each of Problems 9 through 16 find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as \( t \) increases.

9. \( y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1 \)
10. \( y'' + 3y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1 \)
11. \( 6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0 \)
12. \( y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3 \)
13. \( y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0 \)
14. \( 2y'' + y' - 4y = 0, \quad y(0) = 0, \quad y'(0) = 1 \)
15. \( y'' + 3y' - 2y = 0, \quad y(1) = 1, \quad y'(1) = 0 \)
16. \( 4y'' - y = 0, \quad y(-2) = 1, \quad y'(-2) = -1 \)

17. Find a differential equation whose general solution is \( y = c_1 e^{2t} + c_2 e^{-3t} \).
18. Find a differential equation whose general solution is \( y = c_1 e^{-2t} + c_2 e^{2t} \).

19. Find the solution of the initial value problem

\[ y'' - y = 0, \quad y(0) = \frac{1}{2}, \quad y'(0) = -\frac{3}{2}. \]

Plot the solution for \( 0 \leq t \leq 2 \) and determine its minimum value.

20. Find the solution of the initial value problem

\[ 2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \]

Then determine the maximum value of the solution and also find the point where the solution is zero.

21. Solve the initial value problem \( y'' + y' - 2y = 0, y(0) = \alpha, y'(0) = 2 \). Then find \( \alpha \) so that the solution approaches zero as \( t \to \infty \).

22. Solve the initial value problem \( 4y'' - y = 0, y(0) = 2, y'(0) = \beta \). Then find \( \beta \) so that the solution approaches zero as \( t \to \infty \).

In each of Problems 23 and 24 determine the values of \( \alpha \), if any, for which all solutions tend to zero as \( t \to \infty \); also determine the values of \( \alpha \), if any, for which all (nonzero) solutions become unbounded as \( t \to \infty \).

23. \( y'' + (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0 \)
24. \( y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0 \)

25. Consider the initial value problem

\[ 2y'' + 3y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -\beta, \]

where \( \beta > 0 \).

(a) Solve the initial value problem.

(b) Plot the solution when \( \beta = 1 \). Find the coordinates \((t_0, y_0)\) of the minimum point of the solution in this case.

(c) Find the smallest value of \( \beta \) for which the solution has no minimum point.
26. Consider the initial value problem (see Example 5)

\[ y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = \beta, \]

where \( \beta > 0 \).

(a) Solve the initial value problem.

(b) Determine the coordinates \( t_m \) and \( y_m \) of the maximum point of the solution as functions of \( \beta \).

(c) Determine the smallest value of \( \beta \) for which \( y_m \geq 4 \).

(d) Determine the behavior of \( t_m \) and \( y_m \) as \( \beta \to \infty \).

27. Consider the equation \( ay'' + by' + cy = d \), where \( a, b, c, \) and \( d \) are constants.

(a) Find all equilibrium, or constant, solutions of this differential equation.

(b) Let \( y_e \) denote an equilibrium solution, and let \( Y = y - y_e \). Thus \( Y \) is the deviation of a solution \( y \) from an equilibrium solution. Find the differential equation satisfied by \( Y \).

28. Consider the equation \( ay'' + by' + cy = 0 \), where \( a, b, \) and \( c \) are constants with \( a > 0 \). Find conditions on \( a, b, \) and \( c \) such that the roots of the characteristic equation are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

3.2 Solutions of Linear Homogeneous Equations; the Wronskian

In the preceding section we showed how to solve some differential equations of the form

\[ ay'' + by' + cy = 0, \]

where \( a, b, \) and \( c \) are constants. Now we build on those results to provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations. In turn, this understanding will assist us in finding the solutions of other problems that we will encounter later.

To discuss general properties of linear differential equations, it is helpful to introduce a differential operator notation. Let \( p \) and \( q \) be continuous functions on an open interval \( I \), that is, for \( a < t < b \). The cases \( a = -\infty \) or \( b = \infty \), or both, are included. Then, for any function \( \phi \) that is twice differentiable on \( I \), we define the differential operator \( L \) by the equation

\[ L[\phi] = \phi'' + p\phi' + q\phi. \quad (1) \]

Note that \( L[\phi] \) is a function on \( I \). The value of \( L[\phi] \) at a point \( t \) is

\[ L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t). \]

For example, if \( p(t) = t^2, \) \( q(t) = 1 + t, \) and \( \phi(t) = \sin 3t \), then

\[ L[\phi](t) = (\sin 3t)'' + t^2(\sin 3t)' + (1 + t)\sin 3t \]

\[ = -9\sin 3t + 3t^2 \cos 3t + (1 + t)\sin 3t. \]
3.2 Solutions of Linear Homogeneous Equations; the Wronskian

PROBLEMS

In each of Problems 1 through 6 find the Wronskian of the given pair of functions.

1. $e^{t^2}, e^{-3y^2}$
2. $\cos t, \sin t$
3. $e^{-2y}, te^{-2y}$
4. $x, xe^x$
5. $e^t \sin t, e^t \cos t$
6. $\cos^2 \theta, 1 + \cos 2\theta$

In each of Problems 7 through 12 determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. $y'' + 3y = t, \quad y(1) = 1, \quad y'(1) = 2$
8. $(t - 1)y'' - 3y' + 4y = \sin t, \quad y(2) = 2, \quad y'(-2) = 1$
9. $(t - 4)y'' + 3y' + 4y = 2, \quad y(3) = 0, \quad y'(-3) = -1$
10. $y'' + (\cos \theta)y' + 3(\ln \theta)y = 0, \quad y(2) = 3, \quad y'(2) = 1$
11. $(x - 3)y'' + xy' + (2n \ln x)y = 0, \quad y(1) = 0, \quad y'(1) = 1$
12. $(x - 2)y'' + y' + (x - 2)(\tan x)y = 0, \quad y(3) = 1, \quad y'(3) = 2$

13. Verify that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are two solutions of the differential equation $t^2y'' - 2y = 0$ for $t > 0$. Then show that $y = c_1t^2 + c_2t^{-1}$ is also a solution of this equation for any $c_1$ and $c_2$.

14. Verify that $y_1(t) = 1$ and $y_2(t) = e^{t^2}$ are solutions of the differential equation $y'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2e^{t^2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, then $y = c\phi(t)$, where $c$ is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

16. Can $y = \sin(t^2)$ be a solution on an interval containing $t = 0$ of an equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients? Explain your answer.

17. If the Wronskian $W(f, g)$ is $3e^{kt}$, and if $f(t) = e^{kt}$, find $g(t)$.
18. If the Wronskian $W(f, g)$ is $t^2e^t$, and if $f(t) = t$, find $g(t)$.
19. If $W(f, g)$ is the Wronskian of $f$ and $g$, and if $u = 2f - g, v = f + 2g$, find the Wronskian $W(u, v)$ in terms of $W(f, g)$.
20. If the Wronskian of $f$ and $g$ is $t \cos t - \sin t$, and if $u = f + 3g, v = f - g$, find the Wronskian of $u$ and $v$.

21. Assume that $y_1$ and $y_2$ are a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$ and let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where $a_1, a_2, b_1, b_2$ are any constants. Show that

$$W(y_3, y_4) = (a_1b_2 - a_2b_1)W(y_1, y_2).$$

Are $y_3$ and $y_4$ also a fundamental set of solutions? Why or why not?

In each of Problems 22 and 23 find the fundamental set of solutions specified by Theorem 3.2.5 for the given differential equation and initial point.

22. $y'' + y' - 2y = 0, \quad y_0 = 0$
23. $y'' + 4y' + 3y = 0, \quad y_0 = 1$

In each of Problems 24 through 27 verify that the functions $y_1$ and $y_2$ are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

24. $y'' + 4y = 0; \quad y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$
25. $y'' - 2y' + y = 0; \quad y_1(t) = e^t, \quad y_2(t) = te^t$
26. \( x^2 y'' - x(x + 2)y' + (x + 2)y = 0, \quad x > 0; \quad y_1(x) = x, \quad y_2(x) = xe^x \\
27. (1 - x \cos(x))y'' - xy' + y = 0, \quad 0 < x < \pi; \quad y_1(x) = x, \quad y_2(x) = \sin x \\

28. Consider the equation \( y'' - y' - 2y = 0 \).

(a) Show that \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{2t} \) form a fundamental set of solutions.
(b) Let \( y_3(t) = -2e^{2t}, y_4(t) = y_1(t) + 2y_2(t), \) and \( y_5(t) = 2y_1(t) - 2y_2(t) \). Are \( y_3(t), y_4(t), \) and \( y_5(t) \) also solutions of the given differential equation?
(c) Determine whether each of the following pairs forms a fundamental set of solutions: \( y_1(t), y_2(t) \); \( y_3(t), y_5(t) \); \( y_1(t), y_4(t) \); \( y_4(t), y_5(t) \).

In each of Problems 29 through 32 find the Wronskian of two solutions of the given differential equation without solving the equation.

29. \( t^2 y'' - (t + 2)y' + (t + 2)y = 0 \)
30. \( \cos(t)y'' + (\sin t)y' - ty = 0 \)
31. \( x^2 y'' + xy' + (x^2 - y^2)y = 0 \), Bessel's equation
32. \( (1 - x^2)y'' - 2xy' + \alpha(x + 1)y = 0 \), Legendre's equation

33. Show that if \( p \) is differentiable and \( p(t) > 0 \), then the Wronskian \( W(t) \) of two solutions of \( [p(t)y'] + q(t)y = 0 \) is \( W(t) = c/p(t) \), where \( c \) is a constant.

34. If \( y_1 \) and \( y_2 \) are a fundamental set of solutions of \( ty'' + 2y' + te^t y = 0 \) and if \( W(y_1, y_2)(1) = 2 \), find the value of \( W(y_1, y_2)(5) \).

35. If \( y_1 \) and \( y_2 \) are a fundamental set of solutions of \( t^2 y'' - 2y' + (3 + t)y = 0 \) and if \( W(y_1, y_2)(2) = 3 \), find the value of \( W(y_1, y_2)(4) \).

36. If the Wronskian of any two solutions of \( y'' + p(t)y' + q(t)y = 0 \) is constant, what does this imply about the coefficients \( p \) and \( q \)?

37. If \( f, g, \) and \( h \) are differentiable functions, show that \( W(f, h) = f^2 W(g, h) \).

In Problems 38 through 40 assume that \( p \) and \( q \) are continuous and that the functions \( y_1 \) and \( y_2 \) are solutions of the differential equation \( y'' + p(t)y' + q(t)y = 0 \) on an open interval \( I \).

38. Prove that if \( y_1 \) and \( y_2 \) are zero at the same point in \( I \), then they cannot be a fundamental set of solutions on that interval.

39. Prove that if \( y_1 \) and \( y_2 \) have maxima or minima at the same point in \( I \), then they cannot be a fundamental set of solutions on that interval.

40. Prove that if \( y_1 \) and \( y_2 \) have a common point of inflection \( t_0 \) in \( I \), then they cannot be a fundamental set of solutions on \( I \) unless both \( p \) and \( q \) are zero at \( t_0 \).

41. Exact Equations. The equation \( P(x)y'' + Q(x)y' + R(x)y = 0 \) is said to be exact if it can be written in the form \( [P(x)y'] + [f(x)y'] = 0 \), where \( f(x) \) is to be determined in terms of \( P(x), Q(x), \) and \( R(x) \). The latter equation can be integrated once immediately, resulting in a first order linear equation for \( y \) that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating \( f(x) \), show that a necessary condition for exactness is \( P''(x) - Q'(x) + R(x) = 0 \). It can be shown that this is also a sufficient condition.

In each of Problems 42 through 45 use the result of Problem 41 to determine whether the given equation is exact. If so, solve the equation.

42. \( y'' + xy' + y = 0 \)
43. \( y'' + 3x^2 y' + xy = 0 \)

44. \( xy' - (\cos x)y' + (\sin x)y = 0, \quad x > 0 \)
45. \( x^2 y'' + xy' - y = 0, \quad x > 0 \)

46. The Adjoint Equation. If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor \( \mu(x) \). Thus we require that \( \mu(x) \) be such that \( \mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0 \) can be written
3.3 Complex Roots of the Characteristic Equation

Note that if the real part of the roots is zero, as in this example, then there is no exponential factor in the solution. Figure 3.3.3 shows the graph of two typical solutions of Eq. (28). In each case the solution is a pure oscillation whose amplitude is determined by the initial conditions. Since there is no exponential factor in the solution (29), the amplitude of each oscillation remains constant in time.

<table>
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<td>22. ( y'' + 2y' + 2y = 0 ), ( y(\pi/4) = 2 ), ( y'(\pi/4) = -2 )</td>
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23. Consider the initial value problem \( 3u'' - u' + 2u = 0 \), \( u(0) = 2 \), \( u'(0) = 0 \).

(a) Find the solution \( u(t) \) of this problem.

(b) For \( t > 0 \) find the first time at which \( |u(t)| = 10 \).

24. Consider the initial value problem \( 5u'' + 2u' + 7u = 0 \), \( u(0) = 2 \), \( u'(0) = 1 \).

(a) Find the solution \( u(t) \) of this problem.

(b) Find the smallest \( T \) such that \( |u(t)| \leq 0.1 \) for all \( t > T \).

25. Consider the initial value problem \( y'' + 2y' + 6y = 0 \), \( y(0) = 2 \), \( y'(0) = \alpha \geq 0 \).

(a) Find the solution \( y(t) \) of this problem.

(b) Find \( \alpha \) so that \( y = 0 \) when \( t = 1 \).

(c) Find, as a function of \( \alpha \), the smallest positive value of \( t \) for which \( y = 0 \).

(d) Determine the limit of the expression found in part (c) as \( \alpha \to \infty \).
26. Consider the initial value problem

\[ y'' + 2ay' + (a^2 + 1)y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

(a) Find the solution \( y(t) \) of this problem.
(b) For \( a = 1 \) find the smallest \( T \) such that \( |y(t)| < 0.1 \) for \( t > T \).
(c) Repeat part (b) for \( a = 1/4, 1/2, \) and 2.
(d) Using the results of parts (b) and (c), plot \( T \) versus \( a \) and describe the relation between \( T \) and \( a \).

27. Show that \( W(e^{im} \cos \mu t, e^{im} \sin \mu t) = \mu e^{2\pi t} \).

28. In this problem we outline a different derivation of Euler’s formula.

(a) Show that \( y_1(t) = \cos t \) and \( y_2(t) = \sin t \) are a fundamental set of solutions of \( y'' + y = 0 \); that is, show that they are solutions and that their Wronskian is not zero.
(b) Show (formally) that \( y = e^t \) is also a solution of \( y'' + y = 0 \).
Therefore
\[ e^t = c_1 \cos t + c_2 \sin t \]  

for some constants \( c_1 \) and \( c_2 \). Why is this so?
(c) Set \( t = 0 \) in Eq. (i) to show that \( c_1 = 1 \).
(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set \( t = 0 \) to conclude that \( c_2 = 1 \). Use the values of \( c_1 \) and \( c_2 \) in Eq. (i) to arrive at Euler’s formula.

29. Using Euler’s formula, show that
\[ \cos t = (e^t + e^{-t})/2, \quad \sin t = (e^t - e^{-t})/2i. \]

30. If \( e^r \) is given by Eq. (13), show that \( e^{(r+\alpha)t} = e^{rt}e^{\alpha t} \) for any complex numbers \( r_1 \) and \( r_2 \).

31. If \( e^r \) is given by Eq. (13), show that
\[ \frac{d}{dt} e^{rt} = re^{rt} \]

for any complex number \( r \).

32. Let the real-valued functions \( p \) and \( q \) be continuous on the open interval \( I \), and let \( y = \phi(t) = u(t) + iv(t) \) be a complex-valued solution of
\[ y'' + p(t)y' + q(t)y = 0, \]

where \( u \) and \( v \) are real-valued functions. Show that \( u \) and \( v \) are also solutions of Eq. (i).

Hint: Substitute \( y = \phi(t) \) in Eq. (i) and separate into real and imaginary parts.

33. If the functions \( y_1 \) and \( y_2 \) are a fundamental set of solutions of \( y'' + p(t)y' + q(t)y = 0 \), show that between consecutive zeros of \( y_1 \) there is one and only one zero of \( y_2 \). Note that this result is illustrated by the solutions \( y_1(t) = \cos t \) and \( y_2(t) = \sin t \) of the equation \( y'' + y = 0 \).

Hint: Suppose that \( r_1 \) and \( r_2 \) are two zeros of \( y_1 \) between which there are no zeros of \( y_2 \). Apply Rolle’s theorem to \( y_1/y_2 \) to reach a contradiction.

Change of Variables. Sometimes a differential equation with variable coefficients,
\[ y'' + p(t)y' + q(t)y = 0, \]
can be put in a more suitable form for finding a solution by making a change of the independent variable. We explore these ideas in Problems 34 through 46. In particular, in Problem 34 we show that a class of equations known as Euler equations can be transformed into equations with constant coefficients by a simple change of the independent variable. Problems 35 through
3.3 Complex Roots of the Characteristic Equation

42 are examples of this type of equation. Problem 43 determines conditions under which the more general Eq. (i) can be transformed into a differential equation with constant coefficients. Problems 44 through 46 give specific applications of this procedure.

34. Euler Equations. An equation of the form

\[ t^\alpha \frac{d^2 y}{dt^2} + at \frac{dy}{dt} + \beta y = 0, \quad t > 0, \]  

(ii)

where \( \alpha \) and \( \beta \) are real constants, is called an Euler equation.

(a) Let \( x = \ln t \) and calculate \( dy/dt \) and \( d^2 y/dt^2 \) in terms of \( dy/dx \) and \( d^2 y/dx^2 \).

(b) Use the results of part (a) to transform Eq. (ii) into

\[ \frac{d^2 y}{dx^2} + (\alpha - 1) \frac{dy}{dx} + \beta y = 0. \]  

(iii)

Observe that Eq. (iii) has constant coefficients. If \( y_1(x) \) and \( y_2(x) \) form a fundamental set of solutions of Eq. (iii), then \( y_1(\ln t) \) and \( y_2(\ln t) \) form a fundamental set of solutions of Eq. (ii).

In each of Problems 35 through 42 use the method of Problem 34 to solve the given equation for \( t > 0 \).

35. \( t^3 y'' + ty' + y = 0 \)

36. \( t^3 y'' + 4ty' + 2y = 0 \)

37. \( t^3 y'' + 3ty' + 1.25y = 0 \)

38. \( t^3 y'' - 4ty' - 6y = 0 \)

39. \( t^3 y'' - 4ty' + 6y = 0 \)

40. \( t^3 y'' - ty' + 5y = 0 \)

41. \( t^3 y'' + 3ty' - 3y = 0 \)

42. \( t^3 y'' + 7ty' + 10y = 0 \)

43. In this problem we determine conditions on \( p \) and \( q \) that enable Eq. (i) to be transformed into an equation with constant coefficients by a change of the independent variable. Let \( x = u(t) \) be the new independent variable, with the relation between \( x \) and \( t \) to be specified later.

(a) Show that

\[ \frac{dy}{dt} = \frac{dx}{dt} \frac{dy}{dx}, \quad \frac{d^2 y}{dt^2} = \left( \frac{dx}{dt} \right)^2 \frac{d^2 y}{dx^2} + \left( \frac{dx}{dt} \right) \frac{dy}{dx}. \]

(b) Show that the differential equation (i) becomes

\[ \left( \frac{dx}{dt} \right)^2 \frac{d^2 y}{dx^2} + \left( \frac{d^2 x}{dt^2} + p(t) \frac{dx}{dt} \right) \frac{dy}{dx} + q(t)y = 0. \]  

(iv)

(c) In order for Eq. (iv) to have constant coefficients, the coefficients of \( d^2 y/dx^2 \) and of \( y \) must be proportional. If \( q(t) > 0 \), then we can choose the constant of proportionality to be 1; hence

\[ x = u(t) = \int |q(t)|^{1/2} dt. \]  

(v)

(d) With \( x \) chosen as in part (c), show that the coefficient of \( dy/dx \) in Eq. (iv) is also a constant, provided that the expression

\[ \frac{q(t) + 2p(t)q(t)}{2|q(t)|^{1/2}} \]

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function \((q + 2pq)/q^{1/2}\) is a constant. How must this result be modified if \( q(t) < 0 \)?
original second order equation for \( y \). Although it is possible to write down a formula for \( v(t) \), we will instead illustrate how this method works by an example.

**Example 3**

Given that \( y_1(t) = r^{-1} \) is a solution of

\[
2t^2 y'' + 3ty' - y = 0, \quad t > 0,
\]

we find a fundamental set of solutions.

We set \( y = u(t)r^{-1} \); then

\[
y' = u' r^{-1} - u r^{-2}, \quad y'' = u'' r^{-1} - 2u' r^{-2} + 2u r^{-3}.
\]

Substituting for \( y, y' \), and \( y'' \) in Eq. (31) and collecting terms, we obtain

\[
2t^2(u'' r^{-1} - 2u' r^{-2} + 2ur^{-3}) + 3t(u' r^{-1} - ur^{-2}) - vr^{-3} = 2tu'' + (-4 + 3u') r^{-1} + (4r^{-1} - 3r^{-1} - r^{-3})u
\]

\[
= 2tu'' - ur' = 0.
\]

(32)

Note that the coefficient of \( u \) is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq. (32) and solving for \( u'(t) \), we find that

\[
u'(t) = cr^{1/2} ;
\]

then

\[
u(t) = \frac{3}{2} cr^{3/2} + k.
\]

It follows that

\[
y = \frac{3}{2} cr^{3/2} + kr^{-1},
\]

(33)

where \( c \) and \( k \) are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of \( y_1(t) \) and can be dropped, but the first term provides a new solution \( y_2(t) = r^{1/2} \).

You can verify that the Wronskian of \( y_1 \) and \( y_2 \) is

\[
W(y_1, y_2)(t) = \frac{3}{2} t^{-3/2}, \quad t > 0.
\]

(34)

Consequently, \( y_1 \) and \( y_2 \) form a fundamental set of solutions of Eq. (31).

**Problems**

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. \( y'' - 2y' + y = 0 \)
2. \( 9y'' + 6y' + y = 0 \)
3. \( 4y'' - 4y' - 3y = 0 \)
4. \( 4y'' + 12y' + 9y = 0 \)
5. \( y'' - 2y' + 10y = 0 \)
6. \( y'' - 6y' + 9y = 0 \)
7. \( 4y'' + 17y' + 4y = 0 \)
8. \( 16y'' + 24y' + 9y = 0 \)
9. \( 25y'' - 20y' + 4y = 0 \)
10. \( 2y'' + 2y' + y = 0 \)

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing \( t \).

11. \( 9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1 \)
12. \( y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2 \)
13. \( 9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2 \)
14. \( y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1 \)
15. Consider the initial value problem
\[ 4y'' + 12y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = -4. \]
(a) Solve the initial value problem and plot its solution for \(0 \leq t \leq 5\).
(b) Determine where the solution has the value zero.
(c) Determine the coordinates \((t_0, y_0)\) of the minimum point.
(d) Change the second initial condition to \(y'(0) = b\) and find the solution as a function of \(b\). Then find the critical value of \(b\) that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:
\[ y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = b. \]
Find the solution as a function of \(b\) and then determine the critical value of \(b\) that separates solutions that grow positively from those that eventually grow negatively.

17. Consider the initial value problem
\[ 4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2. \]
(a) Solve the initial value problem and plot the solution.
(b) Determine the coordinates \((t_M, y_M)\) of the maximum point.
(c) Change the second initial condition to \(y'(0) = b > 0\) and find the solution as a function of \(b\).
(d) Find the coordinates \((t_M, y_M)\) of the maximum point in terms of \(b\). Describe the dependence of \(t_M\) and \(y_M\) on \(b\) as \(b\) increases.

18. Consider the initial value problem
\[ 9y'' + 12y' + 4y = 0, \quad y(0) = a > 0, \quad y'(0) = -1. \]
(a) Solve the initial value problem.
(b) Find the critical value of \(a\) that separates solutions that become negative from those that are always positive.

19. If the roots of the characteristic equation are real, show that a solution of
\[ ay'' + by' + cy = 0 \]
is either everywhere zero or else can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation \(y'' + 2ay' + a^2y = 0\). Show that the roots of the characteristic equation are \(r_1 = r_2 = -a\), so that one solution of the equation is \(e^{-at}\).
(b) Use Abel's formula [Eq. (22) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is
\[ W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1e^{-2at}, \]
where \(c_1\) is a constant.
(c) Let \(y_1(t) = e^{-at}\) and use the result of part (b) to obtain a differential equation satisfied by a second solution \(y_2(t)\). By solving this equation, show that \(y_2(t) = te^{-at}\).

21. Suppose that \(r_1\) and \(r_2\) are roots of \(ar^2 + br + c = 0\) and that \(r_1 \neq r_2\); then \(\exp(r_1t)\) and \(\exp(r_2t)\) are solutions of the differential equation \(ay'' + by' + cy = 0\). Show that \(\phi(t; r_1, r_2) = (\exp(r_2t) - \exp(r_1t))/(r_2 - r_1)\) is also a solution of the equation for \(r_2 \neq r_1\).
Then think of \( r_1 \) as fixed and use L'Hopital's rule to evaluate the limit of \( \phi(t; r_1, r_2) \) as \( r_2 \to r_1 \), thereby obtaining the second solution in the case of equal roots.

22. (a) If \( ar^2 + br + c = 0 \) has equal roots \( r_1 \), show that

\[
L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^r = a(r - r_1)^2 e^r.
\]  

Since the right side of Eq. (i) is zero when \( r = r_1 \), it follows that \( \exp(r_1 t) \) is a solution of \( L[y] = 0 \).

(b) Differentiate Eq. (i) with respect to \( r \) and interchange differentiation with respect to \( r \) and with respect to \( t \), thus showing that

\[
\frac{\partial}{\partial r} L[e^{rt}] = L \left[ \frac{\partial}{\partial r} e^{rt} \right] = L[e^{rt}] = a(r - r_1)^2 + 2ae^r(r - r_1).
\]  

Since the right side of Eq. (ii) is zero when \( r = r_1 \), conclude that \( t \exp(r_1 t) \) is also a solution of \( L[y] = 0 \).

In each of Problems 23 through 30 use the method of reduction of order to find a second solution of the given differential equation.

23. \( t^2y'' - 4ty' + 6y = 0 \), \( t > 0 \); \( y_1(t) = t^2 \)

24. \( t^2y'' + 2ty' - 2y = 0 \), \( t > 0 \); \( y_1(t) = t \)

25. \( t^2y'' + 2ty' + y = 0 \), \( t > 0 \); \( y_1(t) = t^{-1} \)

26. \( t^2y'' - t(t + 2)y' + (t + 2)y = 0 \), \( t > 0 \); \( y_1(t) = t \)

27. \( xy'' - y + 4x^2y = 0 \), \( x > 0 \); \( y_1(x) = \sin x \)

28. \( (x - 1)y'' - xy' + y = 0 \), \( x > 1 \); \( y_1(x) = e^x \)

29. \( x^2y'' - (x - 0.1875)y = 0 \), \( x > 0 \); \( y_1(x) = x^{1/2} \sin x \)

30. \( x^2y'' + xy' + (x^2 - 0.25)y = 0 \), \( x > 0 \); \( y_1(x) = x^{-1/2} \sin x \)

31. The differential equation

\[
xy'' - (x + N)y' + Ny = 0,
\]

where \( N \) is a nonnegative integer, has been discussed by several authors. One reason why it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is \( y_1(x) = e^x \).

(b) Show that a second solution has the form \( y_2(x) = ce^x \int x^N e^{-x} \, dx \). Calculate \( y_2(x) \) for \( N = 1 \) and \( N = 2 \); convince yourself that, with \( c = -1/N! \),

\[
y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.
\]

Note that \( y_2(x) \) is exactly the first \( N + 1 \) terms in the Taylor series about \( x = 0 \) for \( e^x \), that is, for \( y_1(x) \).

32. The differential equation

\[
y'' + \delta(xy' + y) = 0
\]

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that \( y_1(x) = \exp(-\delta x^2/2) \) is one solution and then find the general solution in the form of an integral.

---

33. The method of Problem 20 can be extended to second order equations with variable coefficients. If $y_1$ is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution $y_2$ satisfies $(y_1 y_2)' = W(y_1, y_2)'y_1^2$, where $W(y_1, y_2)$ is the Wronskian of $y_1$ and $y_2$. Then use Abel's formula [Eq. (22) of Section 3.2] to determine $y_2$.

In each of Problems 34 through 37 use the method of Problem 33 to find a second independent solution of the given equation.

34. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
35. $ty'' - y' + 4t^2y = 0, \quad t > 0; \quad y_1(t) = \sin(t^2)$
36. $(x - 1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
37. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x$

**Behavior of Solutions as $t \to \infty$.** Problems 38 through 40 are concerned with the behavior of solutions as $t \to \infty$.

38. If $a, b$, and $c$ are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

39. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 38 is no longer true, but that all solutions are bounded as $t \to \infty$.

(b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 38 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y'_0$.

40. Show that $y = \sin t$ is a solution of

$$y'' + (k\sin^2 t)y' + (1 - k\cos t\sin t)y = 0$$

for any value of the constant $k$. If $0 < k < 2$, show that $1 - k\cos t\sin t > 0$ and $k\sin^2 t \geq 0$.

Thus observe that even though the coefficients of this variable-coefficient differential equation are nonnegative (and the coefficient of $y'$ is zero only at the points $t = 0, \pi, 2\pi, \ldots$), it has a solution that does not approach zero as $t \to \infty$. Compare this situation with the result of Problem 38. Thus we observe a not unusual situation in the study of differential equations: equations that are apparently very similar can have quite different properties.

**Euler Equations.** In each of Problems 41 through 46 use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

41. $t^2y'' - 3ty' + 4y = 0, \quad t > 0$
42. $t^2y'' + 2ty' + 0.25y = 0, \quad t > 0$
43. $2t^2y'' - 5ty' + 5y = 0, \quad t > 0$
44. $t^2y'' + 3ty' + y = 0, \quad t > 0$
45. $4t^2y'' - 8ty' + 9y = 0, \quad t > 0$
46. $t^2y'' + 5ty' + 13y = 0, \quad t > 0$

### 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad (1)$$

where $p$, $q$, and $g$ are given (continuous) functions on the open interval $I$. The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad (2)$$

...
and we should choose

\[ Y(t) = e^{(α+iβ)t}(A_0t^n + \cdots + A_n) + e^{(α-iβ)t}(B_0t^n + \cdots + B_n), \]

or, equivalently,

\[ Y(t) = e^{αt}(A_0t^n + \cdots + A_n) \cos βt + e^{iαt}(B_0t^n + \cdots + B_n) \sin βt. \]

Usually, the latter form is preferred. If \( α \pm iβ \) satisfy the characteristic equation corresponding to the homogeneous equation, we must, of course, multiply each of the polynomials by \( t \) to increase their degrees by one.

If the nonhomogeneous function involves both \( \cos βt \) and \( \sin βt \), it is usually convenient to treat these terms together, since each one individually may give rise to the same form for a particular solution. For example, if \( g(t) = t \sin t + 2 \cos t \), the form for \( Y(t) \) would be

\[ Y(t) = (A_0t + A_1) \sin t + (B_0t + B_1) \cos t, \]

provided that \( \sin t \) and \( \cos t \) are not solutions of the homogeneous equation.
26. \( y'' + 2y' + 5y = 3e^{-t} \cos 2t - 2te^{-2t} \cos t \)

27. Consider the equation

\[ y'' - 3y' - 4y = 2e^{t} \]  

from Example 5. Recall that \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{2t} \) are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.4), seek a solution of the nonhomogeneous equation of the form \( Y(t) = u(t)y_1(t) = u(t)e^{-t} \), where \( u(t) \) is to be determined.

(a) Substitute \( Y(t) \), \( Y'(t) \), and \( Y''(t) \) into Eq. (i) and show that \( u(t) \) must satisfy \( u'' - 5u' = 2 \).

(b) Let \( w(t) = u'(t) \) and show that \( w(t) \) must satisfy \( w' - 5w = 2 \). Solve this equation for \( w(t) \).

(c) Integrate \( w(t) \) to find \( u(t) \) and then show that

\[ Y(t) = -\frac{1}{2}te^{-t} + \frac{1}{2}e^{2t} + c_2e^{-t} \]

The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of \( t \) and \( e^{-t} \).

28. Determine the general solution of

\[ y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin mx t, \]

where \( \lambda > 0 \) and \( \lambda \neq mx \) for \( m = 1, \ldots, N \).

29. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution \( y(t) = \phi(t) \) of

\[ y'' + y = \begin{cases} 1, & 0 \leq t \leq \pi; \\ \pi e^{-t}, & t > \pi. \end{cases} \]

satisfying the initial conditions \( y(0) = 0 \) and \( y'(0) = 1 \). Assume that \( y \) and \( y' \) are also continuous at \( t = \pi \). Plot the nonhomogeneous term and the solution as functions of time. Hint: First solve the initial value problem for \( t \leq \pi \); then solve for \( t > \pi \), determining the constants in the latter solution from the continuity conditions at \( t = \pi \).

30. Follow the instructions in Problem 29 to solve the differential equation

\[ y'' + 2y' + 5y = \begin{cases} 1, & 0 \leq t \leq \pi/2; \\ 0, & t > \pi/2. \end{cases} \]

with the initial conditions \( y(0) = 0 \) and \( y'(0) = 0 \).

Behavior of Solutions as \( t \to \infty \). In Problems 31 and 32 we continue the discussion started with Problems 38 through 40 of Section 3.4. Consider the differential equation

\[ ay'' + by' + cy = g(t), \]  

where \( a, b, \) and \( c \) are positive.

31. If \( Y_1(t) \) and \( Y_2(t) \) are solutions of Eq. (i), show that \( Y_1(t) - Y_2(t) \to 0 \) as \( t \to \infty \). Is this result true if \( b = 0 \)?

32. If \( g(t) = d, \) a constant, show that every solution of Eq. (i) approaches \( d/c \) as \( t \to \infty \). What happens if \( c = 0 \)? What if \( b = 0 \) also?
Section 1.2, page 15

1. (a) \( y = 5 + (y_0 - 5) e^{-t} \)  
(b) \( y = (5/2) + (y_0 - (5/2)) e^{-2t} \)
(c) \( y = 5 + (y_0 - 5) e^{-2t} \)

Equilibrium solution is \( y = 5 \) in (a) and (c), \( y = 5/2 \) in (b); solution approaches equilibrium faster in (b) and (c) than in (a).

2. (a) \( y = 5 + (y_0 - 5) e^{t} \)  
(b) \( y = (5/2) + (y_0 - (5/2)) e^{2t} \)
(c) \( y = 5 + (y_0 - 5) e^{2t} \)

Equilibrium solution is \( y = 5 \) in (a) and (c), \( y = 5/2 \) in (b); solution diverges from equilibrium faster in (b) and (c) than in (a).

3. (a) \( y = ce^{-at} + (b/a) \)  
(b) \( y = ce^{at} + (b/a) \)
(c) (i) Equilibrium is lower and is approached more rapidly. (ii) Equilibrium is higher. (iii) Equilibrium remains the same and is approached more rapidly.

4. (a) \( y = b/a \)  
(b) \( Y = aY \)

5. (a) \( y_1(t) = ce^{et} \)  
(b) \( y = ce^{at} + (b/a) \)

6. \( y = ce^{-at} + (b/a) \)

7. (a) \( T = 2 \ln 18 \approx 5.78 \text{ months} \)  
(b) \( T = 2 \ln(900/(900 - p_0)) \) months
(c) \( p_0 = 900(1 - e^{-6}) \approx 897.8 \)

8. (a) \( r = (\ln 2)/30 \text{ day}^{-1} \)  
(b) \( r = (\ln 2)/N \text{ day}^{-1} \)

9. (a) \( T = 5 \ln 50 \approx 19.56 \text{ s} \)  
(b) \( T = 718.34 \text{ m} \)

10. (a) \( du/dt = 9.8, \quad v(0) = 0 \)  
(b) \( T = \sqrt{300/49} \approx 7.82 \text{ s} \)
(c) \( v \approx 76.68 \text{ m/s} \)

11. (b) \( v = 49 \tan h(t/5) \text{ m/s} \)
(l) \( T = 9.48 \text{ s} \)

12. (a) \( r \approx 0.0288 \text{ day}^{-1} \)  
(b) \( Q(t) = 100e^{-0.0288t} \)
(c) \( T \approx 24.5 \text{ d} \)

13. 1630 ln(4/3)/ln 2 \approx 672.4 \text{ yr} \)

15. (a) \( u = T + (u_0 - T)e^{-kt} \)  
(b) \( Q(t) \rightarrow CV = Q_c \)

16. 6.69 \text{ h} \)

17. (a) \( Q(t) = CV(1 - e^{-t/(RC)}) \)  
(b) \( Q(t) \rightarrow CV = Q_c \)
(c) \( Q(t) = CV \exp(-(t - t_1)/RC) \)

18. (a) \( Q' = 3(1 - 10^{-4}Q) \), \( Q(0) = 0 \)  
(b) \( Q(t) = 10^3(1 - e^{-3t/10^3}) \), \( t \) in h; after 1 year \( Q \approx 9277.77 \text{ g} \)
(c) \( Q' = -3Q/10^3 \), \( Q(0) = 9277.77 \text{ g} \)
(d) \( Q(t) = 9277.77e^{-3t/10^3} \), \( t \) in h; after 1 year \( Q \approx 670.07 \text{ g} \)
(e) \( T \approx 2.60 \text{ yr} \)

19. (a) \( q' = -q/300, \quad q(0) = 5000 \text{ g} \)  
(b) \( q(t) \approx 5000e^{-t/200} \)
(c) no
(d) \( T = 300 \ln(25/6) \approx 428.13 \text{ min} \approx 7.136 \text{ h} \)

(e) \( r = 250 \ln(25/6) \approx 356.78 \text{ gal/min} \)

Section 1.3, page 24

1. Second order, linear
2. Second order, nonlinear
3. Fourth order, linear
4. First order, nonlinear
5. Second order, nonlinear
6. Third order, linear
15. \( r = -2 \)
16. \( r = \pm 1 \)
17. \( r = 2, -3 \)
18. \( r = 0, 1, 2 \)
19. \( r = -1, -2 \)
20. \( r = 1, 4 \)
21. Second order, linear
22. Second order, nonlinear
23. Fourth order, linear
24. Second order, nonlinear
Chapter 2, page 39

1. (c) $y = ce^{-3t} + (t/3) - (1/9) + e^{2t}$; $y$ is asymptotic to $t/3 - 1/9$ as $t \to \infty$
2. (c) $y = ce^{-t} + 2t^2/3; y \to \infty$ as $t \to \infty$
3. (c) $y = ce^{-t} + 1 + t^2e^{-t}/2; y \to 1$ as $t \to \infty$
4. (c) $y = (c/2)t + (3\cos 2t)/4t + (3\sin 2t)/2; y$ is asymptotic to $(3\sin 2t)/2$ as $t \to \infty$
5. (c) $y = ce^{-t} - 3t; y \to \infty$ or $-\infty$ as $t \to \infty$
6. (c) $y = (c - t\cos t + \sin t)/t^2; y \to 0$ as $t \to \infty$
7. (c) $y = te^{e^{-t}} + ce^{-t}; y \to 0$ as $t \to \infty$
8. (c) $y = (\arctan t + c)/(1 + t^2); y \to 0$ as $t \to \infty$
9. (c) $y = ce^{-t/2} + 3t - 6; y$ is asymptotic to $3t - 6$ as $t \to \infty$
10. (c) $y = -te^{-t} + c; y \to \infty, 0, \text{ or } -\infty$ as $t \to \infty$
11. (c) $y = ce^{-t} + \sin 2t - 2 \cos 2t; y$ is asymptotic to $\sin 2t - 2 \cos 2t$ as $t \to \infty$
12. (c) $y = ce^{-t/2} + 3t^2 - 2t + 24; y$ is asymptotic to $3t^2 - 2t + 24$ as $t \to \infty$
13. $y = 3e^t + 2(t - 1)e^{i\pi/4}$
14. $y = (t^2 - 1)e^{-t/2}$
15. $y = (3e^t - 4e^t + 6e^t + 1)/11t^2$
16. $y = (\sin t)/t^2$
17. $y = (t + 2)e^{2t}$
18. $y = r^{-2}(t^2/4) - 1 - t\cos t + \sin t$
19. $y = -(1 + t)e^{-t/4}, t \neq 0$
20. $y = (t - 1 + 2e^{-t})t, t \neq 0$
21. (b) $y = \frac{1}{2}\cos t + \frac{1}{2}\sin t + (a + \frac{2}{3})e^{2t/3}; a_0 = -\frac{4}{5}$
22. (b) $y = -3e^{2t} + (a + 3)e^{2t^2}; a_0 = -3$
23. (b) $y = 2 + (a(3\pi + 4))e^{2t^3} - 2e^{-2t^2}/(3\pi + 4); a_0 = -2/(3\pi + 4)$
24. (b) $y = te^{-t} + (ea - 1)e^{-t}/t; a_0 = 1/e$
25. (b) $y = -(\cos t / t^2) + \pi^2 a / A^2; a_0 = 4/\pi^2$
26. (b) $y = (e^t - e + a\sin t)/t; a_0 = (e - 1)/\sin 1$
27. $(r,y) = (1.364312, 0.820082)$
28. $y_0 = -1.642876$
29. (b) $y = 12 + \frac{6}{35}\cos 2x + \frac{64}{35}\sin 2x - 288/35 e^{-t/4}$; $y$ oscillates about $12$ as $t \to \infty$
30. $y_0 = -5/2$
31. $y_0 = -16/3; y \to -\infty$ as $t \to \infty$ for $y_0 = -16/3$
39. See Problem 2.
40. See Problem 4.
41. See Problem 6.
42. See Problem 12.

Section 2.2, page 47

1. $3y^2 - 2x^3 = c; y \neq 0$
2. $3y^2 - 2\ln(1 + x^2) = c; x \neq -1, y \neq 0$
3. $y^2 - \cos x = c$ if $y \neq 0$; also $y = 0$ everywhere
4. $3y^2 + y^2 - x^4 = c; y \neq -3/2$
5. $2\tan 2y - 2x - \sin 2x = c$ if $\cos 2y \neq 0$; also $y = (\pm (2n + 1)\pi/4$ for any integer $n$; everywhere
6. $y = \sin(\ln |x| + c)$ if $x \neq 0$ and $|y| < 1$; also $y = \pm 1$
7. $y^2 - x^2 + 2(e^y - e^{-y}) = c; y + e^y \neq 0$
8. $3y^2 - x^3 = c$ everywhere
9. (a) $y = 1/(\sqrt{2x^2 - x - 6})$
10. (a) $y = -\sqrt{2x - 2x^2 + 4}$
11. (a) $y = [2(1 - x)e^x - 1]^{1/2}$
12. (a) $r = 2/(1 - 2\ln \theta)$
13. (a) $y = -[2\ln(1 + x^2) + 4]^{1/2}$
14. (a) \( y = \left[3 - 2\sqrt{1 + x^2}\right]^{-1/3} \)  
(c) \( |x| < \frac{1}{3}\sqrt{3} \)  
15. (a) \( y = -\frac{1}{2} + \frac{1}{3}\sqrt{4x^2 - 15} \)  
(c) \( x > \frac{1}{3}\sqrt{15} \)  
16. (a) \( y = -\left(\sqrt{x^2 + 1}\right)^2 \)  
(c) \( -\infty < x < -\infty \)  
17. (a) \( y = \frac{5}{2} - \sqrt{x^2 - e^x + 13/4} \)  
(c) \(-1.4445 < x < 4.6297\) approximately  
18. (a) \( y = -\frac{1}{2} + \frac{1}{3}\sqrt{65 - 8e^x - 8e^{-x}} \)  
(c) \( |x| < 2.0794\) approximately  
19. (a) \( y = |x - \pi| - \arcsin(3\cos^2 x)/3 \)  
(c) \( |x - \pi/2| < 0.6155 \)  
20. (a) \( y = \left[\frac{1}{2}(\arcsin x)^2 + 1\right]^{1/2} \)  
(c) \( -1 < x < 1 \)  
21. \( y^3 - 3y^2 - x - x^3 + 2 = 0 \)  
22. \( y^2 - 4y - x^2 = -1 \)  
23. \( y = -1/(x^2/2 + 2x - 1); \ x = -2 \)  
24. \( y = -3/2 + \sqrt{2x - e^x + 13/4}; \ x = \ln 2 \)  
25. \( y = -3/2 + \sqrt{\sin 2x + 1/4}; \ x = \pi/4 \)  
26. \( y = \tan((x^2 + 2x); \ x = -1 \)  
27. (a) \( y \to 4 \) if \( y_0 > 0 \); \( y \to 0 \) if \( y_0 = 0 \); \( y \to -\infty \) if \( y_0 < 0 \)  
(b) \( T = 3.29527 \)  
28. (a) \( y \to 4 \) as \( t \to \infty \)  
(b) \( T = 2.84367 \)  
(c) \( 3.6622 < y_0 < 4.4042 \)  
29. \( x = \frac{y + \sqrt{y^2 - 4ac}}{a} \)  
30. (c) \( a \neq 0 \)  
31. (b) \( \text{arctan}(y/x) - \ln |y| = c \)  
32. \( b^2 + 2a > 0 \) \( \text{c} \)  
33. (b) \( |y - x| = c\sqrt{y + 3x^2}; \) also \( y = x \)  
34. \( f(x) = (x+y)^{2x} - (x+y)^{2x} \)  
35. \( f(x) = (x+y)^{2x} - (x+y)^{2x} \)  
36. \( f(x) = (x+y)^{2x} - (x+y)^{2x} \)  
37. (b) \( |x^2| < 5y^2 |= c \)  
38. (b) \( c|x^2| = |y^2 - x^2| \)  

Section 2.3, page 59  
1. \( t = 100 \ln 100 \min \equiv 460.5 \min \)  
2. \( Q(t) = 120y(1 - \exp(-t/60)); \) \( 120y \)  
3. \( Q = 50e^{-0.21}(1 - e^{-0.2}) \text{ lb} \equiv 7.42 \text{ lb} \)  
4. \( Q(t) = 200 + t - (100(200)^2)/(200 + t)^2 \text{ lb}, \ t < 300; \ c = 121/125 \text{ lb/gal}; \) \( \lim_{t \to \infty} c = 1 \text{ lb/gal}; \)  
5. (a) \( Q(t) = \frac{43119}{2501}e^{-t/150} + 25 - \frac{655}{2501}\cos t + \frac{25}{2501}\sin t \)  
6. (c) \( 130.5 \text{ s} \)  
7. (a) \( \frac{9.90}{\text{yr}} \text{ yr} \)  
(b) \( 9.90 \text{ yr} \)  
8. (a) \( k(e^r - 1)/r \)  
(b) \( k \approx \$3930 \)  
(c) \( 9.77\% \)  
9. \( k = \$3086.64\text{yr}^{-1}; \ $1259.92 \)  
10. (a) \$89,034.79 \( \)  
(b) \$102,965.21 \( \)  
11. (a) \( t \approx 135.36 \text{ months} \)  
(b) \$152,698.56 \( \)  
12. (a) \( 0.00012097 \text{ yr}^{-1} \)  
(b) \( Q_0 \exp(-0.00012097t), \ t \text{ in yr} \)  
(c) \( 13.305 \text{ yr} \)  
13. \( P = 201,777.31 - 1977.31e^{0.03t}, \ 0 \leq t \leq \pi \approx 6.6745 \text{ (weeks)} \)  
14. (a) \( \tau \approx 2.9632; \) \( \)  
(b) \( \tau = 10 \ln 2 \approx 6.9315 \)  
(c) \( \tau = 6.3805 \)  
15. (b) \( y_0 \approx 0.83 \)  
16. \( t = \ln \frac{1}{2} / \ln \frac{1}{2} \text{ min} \approx 6.07 \text{ min} \)  
17. (a) \( u(t) = 2000/(1 + 0.0486t)^{1/3} \)  
(c) \( \tau \approx 750.77 \text{ s} \)  
18. (a) \( u(t) = e^{-k^2t} + T_2 + kT_2(k \cos \omega t + \omega \sin \omega t)/(k^2 + \omega^2) \)  
(b) \( R \approx 9.11\text{F}; \ \tau \approx 3.51 \text{ h} \)  
(c) \( R = kT_2/\sqrt{k^2 + \omega^2}; \ \tau = (1/\omega) \arctan(\omega/k) \)
19. (a) \( c = k + (P/r) + \epsilon \sqrt{k - (P/r)} e^{-\alpha t} \); \( \lim_{t \to \infty} c = k + (P/r) \)

(b) \( T = (V \ln 2)/r; \quad T = (V \ln 10)/r \)

(c) Superior, \( T = 431 \text{ yr} \); Michigan, \( T = 71.4 \text{ yr} \); Erie, \( T = 6.05 \text{ yr} \); Ontario, \( T = 17.6 \text{ yr} \)

20. (a) 50.408 m \quad (b) 5.248 s

21. (a) 45.783 m \quad (b) 5.129 s

22. (a) 48.562 m \quad (b) 5.194 s

23. (a) 176.7 ft/s \quad (b) 1074.5 ft \quad (c) 15 ft/s \quad (d) 256.6 ft/s

24. (a) \( dv/dx = -\mu v \quad (b) \mu = (66/25) \ln 10 \text{ mi}^{-1} \approx 6.0788 \text{ mi}^{-1} \)

(c) \( \tau = 900/(11 \ln 10) \text{ s} \approx 35.533 \text{ s} \)

25. (a) \( x_m = -\frac{m^2g}{k^2} \ln \left(1 + \frac{k v_0}{m g} + \frac{m v_0}{k} \right) \quad \iota_m = \frac{m}{k} \ln \left(1 + \frac{k v_0}{m g} \right) \)

26. (a) \( v = -(mg/k) + \left[v_0 + (mg/k) \exp(-kt/m) \right] \quad (b) v = v_0 - gt; \quad \text{yes} \)

(c) \( v = 0 \) for \( t > 0 \)

27. (a) \( v_t = 2\alpha g(\rho - \rho_f)/\mu \quad (b) \quad \epsilon = 4\pi \alpha^2 g(\rho - \rho)/3E \)

28. (a) 11.56 m/s \quad (b) 13.45 m/s \quad (c) \( k \geq 0.2394 \text{ kg/s} \)

29. (a) \( v = R/(2g/R + x) \quad (b) \quad 50.6 \text{ h} \)

30. (a) \( x = u t \cos A, \quad y = -g t^2/2 + u t \sin A + h \quad (b) -16L^2/(u^2 \cos^2 A) + L \tan A + 3 \geq H \)

(c) \( 0.63 \text{ rad} \leq A \leq 0.96 \text{ rad} \)

(f) \( u = 106.89 \text{ ft/s}, \quad A = 0.7954 \text{ rad} \)

31. (a) \( u = (u \cos A)e^{-\alpha t}, \quad \omega = -g/r + (u \sin A + g/r)e^{-\alpha t} \quad (b) x = u \cos A(1 - e^{-\alpha t}), \quad y = -g t/r + (u \sin A + g/r)(1 - e^{-\alpha t})/r + h \quad (d) \quad u = 145.3 \text{ ft/s}, \quad A = 0.644 \text{ rad} \)

32. (d) \( k = 2.193 \)

Section 2.4, page 75

1. \( 0 < t < 3 \)

2. \( x/2 < t < 3x/2 \)

5. \(-2 < t < 2 \)

6. \( 1 < t < x \)

7. \( 2r + 5y > 0 \) or \( 2r + 5y < 0 \)

8. \( r^2 + y^2 < 1 \)

9. \( 1 - r^2 + y^2 > 0 \) or \( 1 - r^2 + y^2 < 0 \), \( t \neq 0, \quad y \neq 0 \)

10. \( \text{Everywhere} \)

11. \( y \neq 0 \), \( y \neq 3 \)

12. \( t \neq n \pi \) for \( n = 0, \pm 1, \pm 2, \ldots \)

13. \( y = \pm(\sqrt{y_0} - 4t^2) \) if \( y_0 \neq 0, \quad |t| < y_0/2 \)

14. \( y = [(1/y_0)^{-2} - 1]^{-1} \) if \( y_0 \neq 0, \quad y = 0 \) if \( y_0 = 0, \quad \text{interval is } |t| < 1/\sqrt{y_0} \text{ if } y_0 > 0 \); \( -\infty < t < \infty \text{ if } y_0 = 0 \)

15. \( y = y_0/\sqrt{2y_0^3 + 1} \) if \( y_0 \neq 0, \quad y = 0 \) if \( y_0 = 0, \quad \text{interval is } -1/2\sqrt{y_0} < t < \infty \text{ if } y_0 \neq 0, \quad -\infty < t < \infty \text{ if } y_0 = 0 \)

16. \( y = \pm(\sqrt{1 + (r^2 + y_0^2)} - [1 - \exp(-3y_0^2/2)])^{1/3} < t < \infty \)

17. \( y \to 3 \text{ if } y_0 > 0, \quad y \to 0 \text{ if } y_0 = 0, \quad y \to -\infty \text{ if } y_0 < 0 \)

18. \( y \to -\infty \text{ if } y_0 < 0, \quad y \to 0 \text{ if } y_0 = 0, \quad y \to -\infty \text{ if } y_0 > 0 \)

19. \( y \to -\infty \text{ if } y_0 < 9 \); \( y \to \infty \text{ if } y_0 > 9 \)

20. \( y \to -\infty \text{ if } y_0 < y_c \approx -0.019; \text{ otherwise } y \text{ is asymptotic to } \sqrt{t - 1} \)

21. (a) No \quad (b) Yes; set \( t_0 = 1/2 \) in Eq. (19) in text.

22. (a) \( y_1(t) \) is a solution for \( t \geq 2; \quad y_2(t) \) is a solution for all \( t \)

(b) \( y_2(t) \) is not continuous at \( (2,-1) \)

26. (a) \( y_1(t) = \frac{1}{\mu(t)} \quad \text{y}_2(t) = \frac{1}{\mu(t)} \int_{t_0}^t \mu(s) g(s) \, ds \)

28. \( y = \pm(4t/(2 + 5e^{2t}))^{1/2} \quad 29. \quad y = r/(k + e^{-\alpha t}) \)

30. \( y = \pm \left[\frac{\mu(t)}{2} \int_{t_0}^t \mu(s) g(s) \, ds + c \right]^{1/2} \), where \( \mu(t) = \exp(2\Gamma \sin t + 2\Gamma t) \)
32. \( y = \frac{1}{2}(1 - e^{-2t}) \) for \( 0 \leq t \leq 1; \quad y = \frac{1}{2}(e^2 - 1)e^{-t} \) for \( t > 1 \)

33. \( y = e^{-2t} \) for \( 0 \leq t \leq 1; \quad y = e^{-(t+1)} \) for \( t > 1 \)

Section 2.5, page 88

1. \( y = 0 \) is unstable
2. \( y = -a/b \) is asymptotically stable, \( y = 0 \) is unstable
3. \( y = 1 \) is asymptotically stable, \( y = 0 \) and \( y = 2 \) are unstable
4. \( y = 0 \) is unstable
5. \( y = 0 \) is asymptotically stable
6. \( y = 0 \) is asymptotically stable
7. \( (c) \quad y = \left( y_0 + (1 - y_0)k\right)/(1 + (1 - y_0)k) \)
8. \( y = 1 \) is semistable
9. \( y = -1 \) is asymptotically stable, \( y = 0 \) is semistable, \( y = 1 \) is unstable
10. \( y = -1 \) and \( y = 1 \) are asymptotically stable, \( y = 0 \) is unstable
11. \( y = 0 \) is asymptotically stable, \( y = b^2/a^2 \) is unstable
12. \( y = 2 \) is asymptotically stable, \( y = 0 \) is semistable, \( y = -2 \) is unstable
13. \( y = 0 \) and \( y = 1 \) are semistable
14. (a) \( \tau = (1/r) \ln 4; \quad 55.452 \) yr
    (b) \( T = (1/r) \ln(\beta/(1 - \alpha)/(1 - \beta)\alpha) \); \( 175.78 \) yr
15. (a) \( y = 0 \) is unstable, \( y = K \) is asymptotically stable
    (b) Concave up for \( 0 < y \leq K/e \), concave down for \( K/e \leq y < K \)
16. (a) \( y = K \exp[(\ln(y_0/K))e^{-\tau}] \)
    (b) \( \gamma(2) \cong 0.7153K \cong 57.6 \times 10^6 \) kg
    (c) \( \tau \cong 2.215 \) yr
17. (b) \( (k/\alpha)/K/a\alpha; \quad yes \)
    (c) \( k/\alpha \leq \pi a^2 \)
18. (b) \( k^2/2(g(a)\alpha)^2 \)
19. (c) \( \gamma = E\gamma = KE[1 - (E/r)] \)
20. (d) \( Y_m = Kr/4 \) for \( E = r/2 \)
21. (a) \( y_1 \cong K(1 + \sqrt{1 - (4\nu/K)})/2 \)
22. (a) \( y = 0 \) is unstable, \( y = 1 \) is asymptotically stable
    (b) \( y = y_0/(y_0 + (1 - y_0)e^{-\nu}) \)
23. (a) \( y = y_0e^{-\nu} \)
    (b) \( x = x_0 \exp[-(h - \alpha y_0(1 - e^{-\nu})/\nu)] \)
    (c) \( x_0 \exp[-(h - \alpha y_0)/\nu] \)
24. (b) \( x = 1/(1 - y + (1 - y)e^{-\nu}) \)
    (c) \( 0.0927 \)
25. (a,b) \( a = 0; \quad y = 0 \) is semistable.
    \( a > 0; \quad y = \sqrt{a} \) is asymptotically stable and \( y = -\sqrt{a} \) is unstable.
26. (a) \( a \leq 0; \quad y = 0 \) is asymptotically stable.
    \( a > 0; \quad y = 0 \) is unstable; \( y = \sqrt{a} \) and \( y = -\sqrt{a} \) are asymptotically stable.
27. (a) \( a < 0; \quad y = 0 \) is asymptotically stable and \( y = a \) is unstable.
    \( a = 0; \quad y = 0 \) is semistable.
    \( a > 0; \quad y = 0 \) is unstable and \( y = a \) is asymptotically stable.

28. (a) \( \lim_{t \to \infty} x(t) = \min(p, q); \quad x(t) = \frac{pq[\exp(qt) - 1]}{q^2 \exp(qt) - p} \)
    (b) \( \lim_{t \to \infty} x(t) = p; \quad x(t) = \frac{p^2 \alpha t}{p \alpha t + 1} \)

Section 2.6, page 99

1. \( x^2 + 3x + y^2 - 2y = c \)
2. Not exact
3. \( x^2 - x^2y + 2x + 2y^3 + 3y = c \)
4. \( x^2y^2 + 2xy = c \)
5. \( ax^2 + 2bxy + cy^2 = k \)
6. Not exact
7. \( e^y \sin y + 2y \cos x = c \); also \( y = 0 \)
8. Not exact
9. \( e^y \cos 2x + x^2 - 3y = c \)
10. \( y \ln x + 3x^2 - 2y = c \)
11. Not exact
12. \( x^2 + y^2 = c \)
13. \( y = \left|x + \sqrt{28 - 3x^2}\right|/2; \quad |x| < \sqrt{28/3} \)
14. \( y = \left|x - (24x^3 + x^2 - 8x - 16)^{1/2}/4\right; \quad x > 0.9846 \)
Answers to Problems

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15. \( b = 3; \quad x^2 + 2xy = c \)
16. \( b = 1; \quad e^{2y} + x^2 = c \)
19. \( x^2 + 2 \ln |y| - y^2 = c; \text{ also } y = 0 \)
20. \( e^y \sin y + 2y \cos x = c \)
21. \( xy^2 - (y^2 - 2y + 2)e^y = c \)
22. \( x^2 \sin y = c \)
24. \( \mu(t) = \exp \int R(t) \, dt, \text{ where } t = xy \)
25. \( \mu(x) = e^{3x}; \quad (3x^2y + y^3)e^{3x} = c \)
26. \( \mu(x) = e^{-x}; \quad y = ce^{-x} + 1 + e^{2x} \)
27. \( \mu(y) = y; \quad xy + y \cos y - \sin y = c \)
28. \( \mu(x) = e^{2y}/y; \quad xe^{y} - \ln |y| = c; \text{ also } y = 0 \)
29. \( \mu(y) = \sin y; \quad e^x \sin y + y^2 = c \)
30. \( \mu(y) = y^3; \quad x^4 + 3xy + y^3 = c \)
31. \( \mu(x,y) = xy; \quad x^3y + 3x^2 + y^3 = c \)

Section 2.7, page 109

1. (a) 1.2, 1.39, 1.571, 1.7439
   (c) 1.19631, 1.38335, 1.5620, 1.73038
   (d) 1.19516, 1.38127, 1.55918, 1.72968
   (b) 1.2, 1.22, 1.364, 1.5368
   (e) 1.10575, 1.23873, 1.39793, 1.59144
   (f) 1.1007, 1.24539, 1.41016, 1.61277
2. (a) 1.25, 1.54, 1.878, 2.2736
   (c) 1.26551, 1.57746, 1.94586, 2.38287
   (d) 1.27142, 1.59182, 1.97212, 2.42554
   (b) 0.3, 0.538501, 0.724821, 0.864658
   (e) 0.284181, 0.313339, 0.693451, 0.831571
   (f) 0.277920, 0.501813, 0.678949, 0.815302
   (g) 0.271428, 0.490897, 0.665142, 0.799729
5. Converge for \( y \geq 0 \); undefined for \( y < 0 \)
6. Converge for \( y \geq 0 \); diverge for \( y < 0 \)
7. Converge
8. Converge for \( |y(0)| < 2.37 \) (approximately); diverge otherwise
9. Diverge
10. Diverge
11. (a) 2.30800, 2.49005, 2.60023, 2.66773, 2.70939, 2.73521
   (b) 2.30167, 2.48263, 2.59352, 2.66227, 2.70519, 2.73209
   (c) 2.29864, 2.47903, 2.59024, 2.65958, 2.70310, 2.73053
   (d) 2.29686, 2.47691, 2.58830, 2.65798, 2.70185, 2.72959
   (e) 1.70308, 3.06605, 2.44010, 1.77204, 1.37348, 1.11925
   (f) 1.79548, 3.06651, 2.43292, 1.77807, 1.37795, 1.12191
   (g) 1.84579, 3.05769, 2.42905, 1.78074, 1.38017, 1.12328
   (h) 1.87734, 3.05607, 2.42672, 1.78224, 1.38150, 1.12411
   (i) 1.48739, 3.04830, 2.42605, 1.77108, 1.36488, 1.10721
   (j) 1.48718, 3.04751, 2.42585, 1.77063, 1.36424, 1.10656
12. (a) -1.48739, -0.412339, 1.04687, 1.43176, 1.54438, 1.51971
   (b) -1.46009, -0.287883, 1.05351, 1.42003, 1.53000, 1.50549
   (c) -1.45865, -0.217545, 1.05715, 1.41846, 1.52334, 1.49879
   (d) -1.45212, -0.173376, 1.05941, 1.41197, 1.51949, 1.49940
13. (a) 0.950517, 0.887550, 0.369188, 0.145990, 0.0421429, 0.00872877
   (b) 0.936298, 0.672345, 0.362640, 0.147359, 0.045100, 0.0104931
   (c) 0.932253, 0.664778, 0.395967, 0.148416, 0.0469514, 0.0113722
   (d) 0.928649, 0.660463, 0.357783, 0.148848, 0.0478492, 0.0118978
14. (a) -0.166134, -0.410872, -0.804660, 4.15867
   (b) -0.174652, -0.434238, -0.889140, -3.09810
15. A reasonable estimate for \( y \) at \( t = 0.8 \) is between 5.5 and 6. No reliable estimate is possible at \( t = 1 \) from the specified data.
16. A reasonable estimate for \( y \) at \( t = 2.5 \) is between 18 and 19. No reliable estimate is possible at \( t = 3 \) from the specified data.
17. \( 2.37 < a_0 < 2.38 \)
18. (b) \( 0.57 < a_0 < 0.68 \)
Section 2.8, page 118

1. \( d\omega/ds = (s + 1)^2 + (w + 2)^2, \quad \omega(0) = 0 \)
2. \( d\omega/ds = 1 - (w + 3)^3, \quad \omega(0) = 0 \)
3. (a) \( \phi_n(t) = \sum_{k=1}^{n} \frac{t^{k-1}}{k!} \)  \( \lim_{n \to \infty} \phi_n(t) = e^t - 1 \)
4. (a) \( \phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^k t^k}{k!} \)  \( \lim_{n \to \infty} \phi_n(t) = e^{-t} - 1 \)
5. (a) \( \phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^{k+1} t^{k+1}}{(k + 1)!} \)  \( \lim_{n \to \infty} \phi_n(t) = 4e^{-2t} + 2t - 4 \)
6. (a) \( \phi_n(t) = t - \frac{t^{n+1}}{(n + 1)!} \)  \( \lim_{n \to \infty} \phi_n(t) = t \)
7. (a) \( \phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k-1}}{1 \cdot 3 \cdot 5 \cdots (2k - 1)} \)  \( \sum_{k=1}^{n} \frac{t^{2k-1}}{2 \cdot 5 \cdots (3k - 1)} \)
8. (a) \( \phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k-1}}{2 \cdot 5 \cdots (3k - 1)} \)
9. (a) \( \phi_1(t) = \frac{t^2}{21} + \frac{t^4}{31} + \frac{t^6}{61} + O(t^8) \)
\( \phi_2(t) = \frac{t^2}{21} + \frac{t^4}{31} + \frac{t^6}{61} + O(t^8) \)
\( \phi_3(t) = \frac{t^2}{21} + \frac{t^4}{31} + \frac{t^6}{61} + O(t^8) \)
\( \phi_4(t) = t^2 + t^4 + \frac{t^6}{8} + \frac{t^8}{60} + O(t^{10}) \)
\( \phi_5(t) = t^2 + t^4 + \frac{t^6}{8} + \frac{t^8}{60} + O(t^{10}) \)
\( \phi_6(t) = t^2 + t^4 + \frac{t^6}{8} + \frac{t^8}{60} + O(t^{10}) \)

Section 2.9, page 130

1. \( y_n = (-1)^n(y_0)^n y_0; \quad y_n \to 0 \text{ as } n \to \infty \)
2. \( y_n = y_0/(n + 1); \quad y_n \to 0 \text{ as } n \to \infty \)
3. \( y_n \to y_0 \sqrt{(n + 2)(n + 1)/2}; \quad y_n \to \infty \text{ as } n \to \infty \)
4. \( y_n = \begin{cases} 
    y_0, & \text{if } n = 4k \text{ or } n = 4k - 1; \\
    -y_0, & \text{if } n = 4k - 2 \text{ or } n = 4k - 3;
\end{cases} \)
\( y_n \) has no limit as \( n \to \infty \)
5. \( y_n = (0.5)^n(y_0 - 12); \quad y_n \to 12 \text{ as } n \to \infty \)
6. \( y_n = (-1)^n(y_0 - 4) + 4; \quad y_n \to 4 \text{ as } n \to \infty \)
7. 7.25%  \( \text{8. } \$2283.63 \)
8. \$258.14
9. \$804.62
10. (a) \$877.57 \quad (b) \$1028.61 \quad (c) \$1208.61
11. 30 years: \$804.62/month; \$289,663.20 total 20 years: \$899.73/month; \$215,935.20 total
12. \$103,624.62
13. 9.73%
Answers to Problems

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Miscellaneous Problems, page 132

1. \[ y := \frac{c}{x^2} + \frac{x^3}{5} \]
2. \[ 2y + \cos y - x - \sin x = c \]
3. \[ x^2 + xy - 3y - y^3 = 0 \]
4. \[ y = -3e^{x^2} + 2 \]
5. \[ x^2y + xy^2 + x = c \]
6. \[ y = \frac{1}{x^3} \frac{e^{x^2}}{3} \frac{e^{-x^2}}{3} \]
7. \[ x^4 - x - y^2 + y^3 = c \]
8. \[ y = (4 + \cos 2 - \cos x)/x^2 \]
9. \[ x^3y + x + y^2 = c \]
10. \[ x + \ln |x| + x^{-1} + y - 2\ln |y| = c; \text{ also } y = 0 \]
11. \[ (x^2)/3 + xy + e^x = c \]
12. \[ y = ce^{x^2} + e^{-x^2} \ln(1 + e^x) \]
13. \[ y = \tan(x + x^3) + c \]
14. \[ x^2 + 2xy + 3y^2 = 34 \]
15. \[ y = c/\cosh^2(x/2) \]
16. \[ e^{-x}\cos y + e^{x^2}\sin x = c \]
17. \[ y = ce^{x^2} - e^{x^2} \]
18. \[ y = -e^{-2x} - e^{-x} \int_0^x e^{s^2} \, ds + 3e^{-2x} \]
19. \[ 2xy + x^2 - x^3 = c \]
20. \[ e^x + e^{-x} = c \]
21. \[ 2xy^2 + 3x^2y - 4x + y^2 = c \]
22. \[ y^3 + 3y - x^3 + 3x = 2 \]
23. \[ y = e^{x^2} + e^{-x^2} \]
24. \[ \sin y \sin x = c \]
25. \[ (x^2 + y^2) + \arctan(y/x) = c \]
26. \[ e^{-y^2} + \ln |x| = c \]
27. \[ (x^2 + y^2 + 1)e^{y^2} = c \]
28. \[ x^3 + x^2y = c \]
29. \[ \arctan(y/x) - \ln \sqrt{x^2 + y^2} = c \]
30. \[ (y^2/x^2) + (y/x^2) = c \]
31. \[ x^2y^2 + x^3 = -4 \]
32. \[ \frac{1}{y} = \int_1^{e^t} e^{s^2} \, ds + \frac{x}{2} \]
33. \[ (a) \quad y = t + (c - t)^{-1} \]
34. \[ (b) \quad y = \sin t + (c \cos t - \frac{1}{2} \sin t)^{-1} \]
35. \[ (c) \quad y = \sin t + (c \cos t - \frac{1}{2} \sin t)^{-1} \]
36. \[ (b) \quad y = \frac{1}{a} \int \mu(t) \, dt + c \]
37. \[ \mu(t) = \exp(-at^2/2) - bt \]
38. \[ y = (y/k) \ln |k - t|/(k + t) + c_2 \text{ if } c_1 = k^2 > 0; \quad y = (2/k) \arctan(t/k) + c_1 \text{ if } c_1 = -k^2 < 0; \]
39. \[ y = \pm \sqrt{(t - 2c_1)/4} + c_1 + c_2; \text{ also } y = c \]
40. \[ y = \frac{1}{2} \ln(1 + 1/n) + c \]
41. \[ c_1 e^{x^2} \ln |x| + c_1 t + c_2 \text{ if } c_1 \neq 0; \quad y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 = 0; \text{ also } y = c \]
42. \[ y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 \neq 0; \quad y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 = 0; \text{ also } y = c \]
43. \[ y = c_0 \sin t + c_2 \cos t \]
44. \[ y = \frac{1}{2} \ln(1 + 1/n) + c \]
45. \[ y = \frac{1}{2} \ln(1 + 1/n) + c \]
46. \[ y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 \neq 0; \quad y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 = 0; \text{ also } y = c \]
47. \[ y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 \neq 0; \quad y = \frac{1}{2} e^{x^2} + c_2 \text{ if } c_1 = 0; \text{ also } y = c \]
48. \[ y = \frac{1}{2} (t + 1)^{3/2} - \frac{1}{2} \]
49. \[ y = \frac{1}{2} \ln(1 + 1/n) + c \]
50. \[ y = 3 \ln t + \frac{1}{2} \ln(1 + 1/n) - 5 \arctan t + 2 + \frac{1}{2} \ln 2 + \frac{1}{2} n \]
51. \[ y = \frac{1}{2} t^2 + \frac{1}{2} \]

Section 3.1, page 144

1. \[ y = c_0 e^t + c_2 e^{-3t} \]
2. \[ y = c_0 e^t + c_2 e^{-3t} \]
3. \[ y = c_0 e^{t^2} + c_2 e^{-t^3} \]
4. \[ y = c_0 e^{t^2} + c_2 e^{-t^3} \]
5. \[ y = c_0 + c_2 e^{-3t} \]
6. \[ y = c_0 e^{t^2} + c_2 e^{-3t/2} \]
7. \[ y = c_1 \exp[9 + 3\sqrt{5}t]/2 + c_2 \exp[9 - 3\sqrt{5}t]/2 \]
8. \[ y = c_1 \exp[1 + \sqrt{3}t] + c_2 \exp[1 - \sqrt{3}t] \]
9. \[ y = e^t; \quad y \to \infty \text{ as } t \to \infty \]
10. \[ y = \frac{1}{2} e^{t^2} - \frac{1}{2} e^{-t^2}; \quad y \to 0 \text{ as } t \to \infty \]
11. \[ y = 12e^{t^2} - 8e^{-t^2}; \quad y \to -\infty \text{ as } t \to \infty \]
12. \[ y = -1 - e^{-3t}; \quad y \to -1 \text{ as } t \to \infty \]
13. \[ y = \frac{1}{2}(13 + 5\sqrt{13}) \exp([-5 + \sqrt{13}t]/2) + \frac{1}{2}(13 - 5\sqrt{13}) \exp([-5 - \sqrt{13}t]/2); \quad y \to 0 \text{ as } t \to \infty \]
14. \( y = \frac{2}{\sqrt{3}} \exp((-1 + \sqrt{3})t/4) - \frac{2}{\sqrt{3}} \exp((-1 - \sqrt{3})t/4); \ y \to \infty \text{ as } t \to \infty \)
15. \( y = \frac{e}{10} e^{-\alpha t} + \frac{9}{10} e^{-\beta t}; \ y \to \infty \text{ as } t \to \infty \)
16. \( y = -\frac{1}{2} e^{(1+2)a/2} + \frac{3}{2} e^{(-1+2)/a}; \ y \to -\infty \text{ as } t \to \infty \)
17. \( y' + y' - 6y = 0 \quad 18. 2y'' + 5y' + 2y = 0 \)
19. \( y = \frac{1}{4} e^{\alpha t} + e^{-\beta t}; \text{ minimum is } y = 1 \text{ at } t = \ln 2 \)
20. \( y = -\frac{3}{2} e^{2t}; \text{ maximum is } y = \frac{3}{2} \text{ at } t = \ln(9/4), y = 0 \text{ at } t = \ln 9 \)
21. \( a = -2 \quad 22. \beta = -1 \)
23. \( y \to 0 \text{ for } \alpha < 0; \ y \text{ becomes unbounded for } \alpha > 1 \)
24. \( y \to 0 \text{ for } \alpha < 1; \text{ there is no } \alpha \text{ for which all nonzero solutions become unbounded} \)
25. (a) \( y = \frac{1}{2} (1 + 2\beta) e^{3t} + \frac{1}{2} (4 - 2\beta) e^{-2t} \)
    (b) \( y \equiv 0.71548 \text{ when } t = \frac{1}{2} \ln 6 \equiv 0.71670 \quad (c) \beta = 2 \)
26. (a) \( y = (6 + \beta) e^{2t} - (4 + \beta) e^{-3t} \)
    (b) \( t_m = \ln(12 + 3\beta)/(12 + 2\beta), \ y_m = \frac{4}{21} (6 + \beta)^2/(4 + \beta)^2 \)
    (c) \( \beta = 6(1 + \sqrt{3}) \equiv 16.3923 \quad (d) \ t_m \to \infty, \ y_m \to 0 \)
27. (a) \( y = \frac{d}{c} \quad (b) \alpha Y'' + b Y' + c Y = 0 \)
28. (a) \( b > 0 \) and \( 0 < c < b^2/4a \) \quad (b) \( c < 0 \) \quad (c) \( b < 0 \) and \( 0 < c < b^2/4a \)

Section 3.2, page 155

1. \(-\frac{7}{2} e^{t^2/2} \quad 2. 1 \)
2. \( e^{4t} \quad 3. e^{4t} \quad 4. x^2 e^x \)
5. \(-e^{2t} \quad 6. 0 \)
7. \( 0 < t < \infty \quad 8. -\infty < t < 1 \)
9. \( 0 < t < 4 \quad 10. 0 < t < \infty \)
11. \( 0 < x < 3 \quad 12. 2 < x < 3\pi/2 \)
14. The equation is nonlinear. \quad 15. The equation is nonhomogeneous.
16. No \quad 17. 3e^{2t} + ce^{2t} \quad 18. te^t + ct \quad 19. Sw(f, g)
20. \(-4 (t \cos t - \sin t) \quad 21. y_3 \text{ and } y_4 \text{ are a fundamental set of solutions if and only if } a_1 b_2 - a_2 b_1 \neq 0 \)
22. \( y_1(t) = \frac{1}{2} e^{-2t} + \frac{3}{2} e^t, \ y_2(t) = -\frac{1}{2} e^{-2t} + \frac{3}{2} e^t \)
23. \( y_1(t) = -\frac{1}{2} e^{3(t-1)} + \frac{3}{2} e^{-(t-1)}, \ y_2(t) = -\frac{1}{2} e^{3(t-1)} + \frac{3}{2} e^{-(t-1)} \)
24. Yes \quad 25. Yes \quad 26. Yes \quad 27. Yes \quad 28. (b) \quad \textit{Yes} \quad (c) [y_1(t), y_3(t)] \text{ and } [y_1(t), y_4(t)] \text{ are fundamental sets of solutions; } [y_2(t), y_3(t)] \text{ and } [y_4(t), y_5(t)] \text{ are not} \)
29. \( c t^2 e^t \quad 30. c \cos t \)
31. \( c / x \quad 32. c/(1 - x^2) \quad 33. 2/25 \quad 34. c \quad 35. 3\sqrt{2} \equiv 4.946 \)
36. \( p(t) = 0 \text{ for all } t \)
40. If \( t_0 \) is an inflection point, and \( y = \phi(t) \) is a solution, then from the differential equation \( p(t_0) \phi(t_0) + q(t_0) \phi(t_0) = 0 \).
42. Yes, \( y = c_1 e^{x^2/2} \int_0^x e^{x^2/2} \, dt + c_2 e^{-x^2/2} \)
43. No \quad 44. Yes, \( y = \frac{1}{\mu(x)} \left[ c_1 \int_0^x \mu(t) \, dt + c_2 \right], \mu(x) = \exp \left[ - \int \left( \frac{1}{x} + \frac{\cos x}{x} \right) \, dx \right] \)
45. Yes, \( y = c_1 x^{-1} + c_2 x \quad 47. x^2 \mu'' + 3x \mu' + (1 + x^2 - v^2) \mu = 0 \)
48. \( (1 - x^2) \mu'' - 2x \mu' + (\alpha + 1) \mu = 0 \quad 49. \mu'' - x \mu = 0 \)
51. The Legendre and Airy equations are self-adjoint.
Section 3.3, page 163

1. \( e \cos 2 + ie \sin 2 \equiv -1.1312 + 2.4717i \)
2. \( e^{2} \cos 3 - ie^{2} \sin 3 \equiv -7.3151 - 1.0427i \)
3. \(-1\)
4. \( e^{2} \cos(\pi/2) - ie^{2} \sin(\pi/2) = -e^{2}i \equiv -7.3891i \)
5. \( 2 \cos(2n) - 2i \sin(2n) \equiv 1.5385 - 1.2773i \)
6. \( \pi^{-3} \cos(2 \ln \pi) + i \pi^{-3} \sin(2 \ln \pi) \equiv -0.20957 + 0.23959i \)
7. \( y = c_{1} e^t \cos t + c_{2} e^t \sin t \)
8. \( y = c_{1} e^t \cos \sqrt{5}t + c_{2} e^t \sin \sqrt{5}t \)
9. \( y = c_{1} e^{-t} + c_{2} e^{-t} \)
10. \( y = c_{1} e^{-t} \cos t + c_{2} e^{-t} \sin t \)
11. \( y = c_{1} e^{-3t} \cos 2t + c_{2} e^{-3t} \sin 2t \)
12. \( y = c_{1} \cos(3t/2) + c_{2} \sin(3t/2) \)
13. \( y = c_{1} e^{-t/2} \cos(t/2) + c_{2} e^{-t/2} \sin(t/2) \)
14. \( y = c_{1} e^{-t/2} + c_{2} e^{-t/2} \)
15. \( y = c_{1} e^{-t/2} \cos(t/2) + c_{2} e^{-t/2} \sin(t/2) \)
16. \( y = c_{1} e^{-t/2} \cos(3t/2) + c_{2} e^{-t/2} \sin(3t/2) \)
17. \( y = \frac{1}{2} \sin 2t; \) steady oscillation
18. \( y = e^{-2t} \cos t + 2e^{-2t} \sin t; \) decaying oscillation
19. \( y = -e^{-\pi/2} \sin 2t; \) growing oscillation
20. \( y = (1 + 2 \sqrt{3}) \cos t - (2 - \sqrt{3}) \sin t; \) steady oscillation
21. \( y = 2e^{-t/2} \cos t + \frac{1}{2} e^{-t/2} \sin t; \) decaying oscillation
22. \( y = \sqrt{2} e^{-\pi/4} \cos t + \sqrt{2} e^{-\pi/4} \sin t; \) decaying oscillation
23. (a) \( u = 2e^{16 \cos(\sqrt{23}t/6) + (2/\sqrt{23})e^{16 \sin(\sqrt{23}t/6)} \)
(b) \( t = 10.7598 \)
24. (a) \( u = 2e^{-\pi/2} \cos(\sqrt{34}t/5) + (\sqrt{34}e^{-\pi/2} \sin(\sqrt{34}t/5) \)
(b) \( T = 14.5115 \)
25. (a) \( y = 2e^{-t} \cos \sqrt{5}t + [(\alpha + 2)/\sqrt{5}]e^{-t} \sin \sqrt{5}t \)
(b) \( \alpha = 1.50878 \)
(c) \( t = (\pi - \arctan(2\sqrt{5} \sec \alpha))/\sqrt{5} \)
(d) \( \pi/\sqrt{5} \)
26. (a) \( y = e^{-t} \cos t + ae^{-t} \sin t \)
(b) \( T = 1.8763 \)
(c) \( \alpha = \frac{1}{2}, T = 7.5328, \alpha = \frac{3}{4}, T = 4.3003; \alpha = 2, T = 1.5116 \)
27. \( y = c_{1} \cos(\ln t) + c_{2} \sin(\ln t) \)
28. \( y = c_{1} t^{-1} \cos(\ln t) + c_{2} t^{-1} \sin(\ln t) \)
29. \( y = c_{1} t^{-1} + c_{2} t^{-1} \)
30. \( y = c_{1} t^{-3} \cos(\ln t) + c_{2} t^{-3} \sin(\ln t) \)
31. \( y = c_{1} \cos x + c_{2} \sin x, x = \int e^{\frac{-t^{2}}{2}} dt \)
32. Yes, \( y = c_{1} e^{\frac{-t^{2}}{4}} \cos(\sqrt{3} \frac{t^{2}}{4}) + c_{2} e^{\frac{-t^{2}}{4}} \sin(\sqrt{3} \frac{t^{2}}{4}) \)

Section 3.4, page 171

1. \( y = c_{1} e^{-t/2} + c_{2} te^{t/2} \)
2. \( y = c_{1} e^{-t/2} + c_{2} te^{t/2} \)
3. \( y = c_{1} e^{-t/2} + c_{2} e^{\frac{3t}{2}} \)
4. \( y = c_{1} e^{3t/2} + c_{2} e^{-3t/2} \)
5. \( y = c_{1} e^{-t/2} \cos t + c_{2} e^{-t/2} \sin 3t \)
6. \( y = c_{1} e^{3t/2} + c_{2} e^{3t/2} \)
7. \( y = c_{1} e^{-t/2} + c_{2} e^{t/2} \)
8. \( y = c_{1} e^{-3t/2} + c_{2} e^{3t/2} \)
9. \( y = c_{1} \cos(\ln t) + c_{2} \sin(\ln t) \)
10. \( y = e^{-t/2} \cos(\ln t) + c_{2} e^{-t/2} \sin(\ln t) \)
11. \( y = 2e^{2t/3} - \frac{1}{2} t e^{2t/3}, \quad y \rightarrow -\infty \text{ as } t \rightarrow \infty \)
12. \( y = 2te^{3t}, \quad y \rightarrow \infty \text{ as } t \rightarrow \infty \)
13. \( y = -e^{-t/2} \cos 3t + \frac{1}{2} e^{-t/2} \sin 3t, \quad y \rightarrow 0 \text{ as } t \rightarrow \infty \)
14. \( y = 7e^{-2t/3} + 5e^{-2t/3}, \quad y \rightarrow 0 \text{ as } t \rightarrow \infty \)
15. (a) \( y = e^{-t/2} - \frac{1}{2} t e^{-t/2} \)
(b) \( t = \frac{3}{2} \)
(c) \( t_{0} = 16/15, \quad y_{0} = -\frac{3}{2} e^{-t/2} \equiv -0.33649 \)
(d) \( y = e^{-t/2} + (b + \frac{1}{2}) t e^{-t/2}, \quad b = -\frac{3}{2} \)
16. \( y = 2e^{t/3} - (b - 1) t e^{t/3}, \quad b = 1 \)
17. (a) \( y = e^{-t/2} + \frac{1}{2} t e^{-t/2} \)
(b) \( t_{M} = \frac{3}{2}, \quad y_{M} = 5e^{-t/2} \equiv 2.24664 \)
(c) \( y = e^{-t/2} + (b + \frac{1}{2}) t e^{-t/2} \)
(d) \( t_{M} = 4b/(1 + 2b) \rightarrow 2 \text{ as } b \rightarrow \infty; \quad y_{M} = (1 + 2b) \exp[-2b/(1 + 2b)] \rightarrow \infty \) as \( b \rightarrow \infty \)
18. (a) \( y = ae^{-\beta t} + \left( \frac{1}{2} \alpha - 1 \right) \beta e^{-\beta t} \)  
(b) \( a = \frac{3}{5} \)

23. \( y_2(t) = t^2 \)

24. \( y_2(t) = t^{-2} \)

25. \( y_2(t) = r^{-1} \ln t \)

26. \( y_2(t) = te^t \)

27. \( y_2(x) = \cos x \)

28. \( y_2(x) = x \)

29. \( y_2(x) = x \left( 1 - e^{-\frac{x}{a}} \right) \)

30. \( y_2(x) = x \left( \frac{1}{4} \right) \cos x \)

32. \( y = c_1 e^{-\beta t} + c_2 e^{-\beta t} \int_0^t e^{\beta s} \, ds + c_3 e^{-\beta t} \int_0^t e^{\beta s} \, ds \)

33. \( y = y_1(t) \int_0^t y_1(s) \exp \left( - \int_0^t p(r) \, dr \right) \, ds \)

34. \( y_2(t) = r^{-1} \ln t \)

35. \( y_2(t) = \cos x \)

36. \( y_2(x) = x \)

37. \( y_2(x) = x^{-\frac{1}{2}} \cos x \)

39. \( y_0 + (a/b) \gamma_0 \)

41. \( y = c_1 t^2 + c_2 t^2 \ln t \)

42. \( y = c_1 r^{-\frac{3}{2}} + c_2 t^{-\frac{3}{2}} \ln t \)

43. \( y = c_1 t + c_2 t^2 \)

45. \( y = c_1 r^{-\frac{3}{2}} + c_2 t^{-\frac{3}{2}} \ln t \)

46. \( y = c_1 r^{-\frac{3}{2}} \cos(3 \ln t) + c_2 r^{-\frac{3}{2}} \sin(3 \ln t) \)

Section 3.5, page 183

1. \( y = c_1 e^{2t} + c_2 e^{-t} - e^t \)

2. \( y = c_1 e^{t} \cos 2t + c_2 e^{-t} \sin 2t + \frac{3}{2} \sin 2t - \frac{13}{8} \cos 2t \)

3. \( y = c_1 e^{2t} e^{2t} + c_2 e^{-t} e^{-t} + \frac{3}{2} e^{2t} e^{-t} \)

4. \( y = c_1 + c_2 e^{-2t} + \frac{3}{2} t - \frac{1}{2} \sin 2t - \frac{1}{2} \cos 2t \)

5. \( y = c_1 \cos 3t + c_2 \sin 3t + \frac{1}{30} \left( 9 t^2 - 6 t + 1 \right) e^{3t} + \frac{3}{10} \)

6. \( y = c_1 e^{-t} + c_2 e^{-t} + t^2 e^{-t} \)

7. \( y = c_1 e^{-t} + c_2 e^{-t} + t^2 - 6 t + 14 - \frac{1}{10} \sin t - \frac{9}{10} \cos t \)

8. \( y = c_1 \cos t + c_2 \sin t - \frac{1}{2} \cos 2t - \frac{1}{3} \sin 2t \)

9. \( u = c_1 \cos \omega t + c_2 \sin \omega t + \left( \frac{1}{2} a \omega \right) \sin \omega t \)

10. \( u = c_1 \cos \omega t + c_2 \sin \omega t + \left( \frac{1}{2} a \omega \right) \sin \omega t \)

11. \( y = c_1 e^{\omega t} \cos \frac{1}{2} (\omega t) + c_2 e^{\omega t} \sin \left( \frac{1}{2} (\omega t) + \frac{1}{2} \omega t \right) - \frac{1}{4} e^{-\omega t} \)

12. \( y = c_1 e^{-\omega t} + c_2 e^{\omega t} + \frac{1}{2} (\omega t) + \frac{1}{8} e^{-\omega t} \)

13. \( y = e^t - \frac{1}{2} e^{-t} - t - \frac{1}{2} \)

14. \( y = \frac{1}{10} \sin 2t - \frac{1}{30} \cos 2t + \frac{1}{18} t^2 + \frac{1}{6} e^t - \frac{1}{4} + \frac{1}{4} \)

15. \( y = 2 e^t - 3 e^t + \frac{1}{2} t^2 + 4 \)

16. \( y = e^t - \frac{3}{2} e^t + \frac{1}{8} e^{-t} \)

17. \( y = 2 \cos 2t - \frac{1}{8} \sin 2t - \frac{1}{2} \cos 2t \)

18. \( y = e^t \cos 2t + \frac{1}{2} e^t \sin 2t + t e^{-t} \sin 2t \)

19. \( Y(t) = (A_0 t^2 + A_1 t + A_2 t + A_3 t + A_4) + t(B_0 t^2 + B_1 t + B_2) e^{-t} + D \sin 3t + E \cos 3t \)

(b) \( A_0 = 2/15, A_1 = -2/9, A_2 = 8/27, A_3 = -8/27, A_4 = 16/81, B_0 = -1/9, B_1 = -1/9, B_2 = -2/27, D = -1/18, E = -1/18 \)

20. (a) \( Y(t) = (A_0 t^2 + A_1 + A_2 t + (D_0 t^2 + D_1) t) \cos t \)

(b) \( A_0 = 1/2, A_1 = 1, A_2 = 3/4, B_0 = 2/3, B_1 = 1, D_0 = -1/4, D_1 = 0 \)

21. (a) \( Y(t) = \epsilon^2 (A \cos 2t + B \sin 2t) + (D_0 t + D_1) \epsilon^2 \sin t + (E_0 + E_1) \epsilon^2 t \)

(b) \( A = -1/10, B = -3/20, D_0 = -3/2, D_1 = -3, E_0 = 3/2, E_1 = 1/2 \)

22. (a) \( Y(t) = A e^{t} + t(B_0 t^2 + B_1 t + B_2) e^{-t} \cos t + t(D_0 t^2 + D_1 t + D_2) e^{-t} \sin t \)

(b) \( A = 3, B_0 = -2/3, B_1 = 0, B_2 = 1, D_0 = 0, D_1 = 1, D_2 = 1 \)

23. (a) \( Y(t) = A_0 t^2 + A_1 t + A_2 t^2 + t(B_0 t^2 + B_1 t + B_2) e^{2t} \)

(b) \( A_0 = 1/2, A_1 = 1, A_2 = 3/4, B_0 = 2/3, B_1 = 1, D_0 = 0, D_1 = -1/16, E_0 = 1/8, E_1 = 1/16 \)

24. (a) \( Y(t) = (A_0 t^2 + A_1 + A_2) t \sin 2t + t(B_0 t^2 + B_1 t + B_2) \cos 2t \)

(b) \( A_0 = 0, A_1 = 13/16, A_2 = 7/4, B_0 = -1/12, B_1 = 0, B_2 = 13/32 \)

25. (a) \( Y(t) = (A_0 t^2 + A_1 t + A_2) \epsilon \sin 2t + (B_0 t^2 + B_1 t + B_2) \epsilon \cos 2t + e^{-t} (D \cos t + E \sin t) + F e^{-t} \)

(b) \( A_0 = 1/52, A_1 = 10/169, A_2 = -1233/35,152, B_0 = -5/52, B_1 = 73/676, B_2 = -4,015/35,152, D = -3/2, E = 3/2, F = 2/3 \)

26. (a) \( Y(t) = (A_0 t^2 + A_1) e^{-t} \cos 2t + t(B_0 t^2 + B_1) e^{-t} \sin 2t + (D_0 t + D_1) e^{-2 t} \cos t + (E_0 + E_1) e^{-t} \sin t \)

(b) \( A_0 = 0, A_1 = 3/16, B_0 = 3/8, B_1 = 0, D_0 = -2/5, D_1 = -7/25, E_0 = 1/5, E_1 = 1/25 \)
27. (b) \( w = -\frac{3}{5} + c_1 e^t \)

28. \( y = c_1 \cos xt + c_2 \sin xt + \sum_{n=1}^{N} \left[ a_m (\lambda^2 - m^2 \pi^2) \right] \sin m\pi t \)

29. \( y = \begin{cases} 
-\left( 1 + \frac{9}{10} e^{-t} \right) \sin 2t - \frac{1}{2} e^{-t} \cos 2t, & 0 \leq t \leq \pi \\
\frac{1}{5} - \frac{1}{10} e^{-t} \sin 2t - \frac{1}{2} e^{-t} \cos 2t, & \pi < t \leq \frac{\pi}{2} \\
-\left( 1 + e^{t/2} \right) e^{-t} \sin 2t - \frac{1}{10} \left( 1 + e^{t/2} \right) e^{-t} \sin 2t, & \frac{\pi}{2} < t \leq \pi/2 
\end{cases} \)

30. \( y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{8} e^{2t} \)

34. \( y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{8} e^{2t} \)

Section 3.6, page 189

1. \( Y(t) = e^t \)

2. \( Y(t) = -\frac{1}{2} te^{-t} \)

3. \( Y(t) = \frac{1}{2} t^2 e^{-t} \)

4. \( Y(t) = 2te^{t/2} \)

5. \( y = c_1 \cos t + c_2 \sin t - (\cos t) \ln(\tan t + \sec t) \)

6. \( y = c_1 \cos 3t + c_2 \sin 3t + (\sin 3t) \ln(\tan 3t + \sec 3t) - 1 \)

7. \( y = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t} \ln t \)

8. \( y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4} (\sin 2t) \ln 2t - \frac{1}{2} t \cos 2t \)

9. \( y = c_1 \cos(t/2) + c_2 \sin(t/2) + t \sin(t/2) + 2 \ln(\cos(t/2)) \cos(t/2) \)

10. \( y = c_1 e^{e^t} + c_2 e^{-t} - \frac{1}{8} e^{2t} \ln(1 + e^t) + e^t \arctan t \)

11. \( y = c_1 e^{2t} + c_2 e^{-t} + \int \left[ e^{2t-s} - e^{2t-2s} \right] g(s) \, ds \)

12. \( y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{2} \int [\sin 2(t-s)] g(s) \, ds \)

13. \( Y(t) = \frac{1}{2} + \frac{1}{12} \ln t \)

14. \( Y(t) = -2t^2 \)

15. \( Y(t) = \frac{1}{4} (t - 1)^2 e^t \)

16. \( Y(t) = -\frac{1}{2} (2t - 1) e^{-t} \)

17. \( Y(x) = \frac{1}{4} x^2 \ln(x)^3 \)

18. \( Y(x) = -\frac{1}{2} x^{1/2} \cos x \)

19. \( Y(x) = \int \frac{2e^x - xe^x}{1 - e^x} g(t) \, dt \)

20. \( Y(x) = x^{-1/2} \int \frac{1}{(1 - t)^2} \sin(x-t) g(t) \, dt \)

23. (b) \( y = y_0 \cos t + y_0 \sin t + \int_0^t \sin(t-s) g(s) \, ds \)

24. \( y = (b - a)^{-1} \int_0^t [e^{t-s} - e^{a-t}] g(s) \, ds \)

25. \( y = \mu^{-1} \int_0^t e^{t-s} \sin \mu(t-s) g(s) \, ds \)

26. \( y = \int_{y_0}^t (t-s) e^{a-t-s} g(s) \, ds \)

29. \( y = c_1 e^t + c_2 e^{3t} + 4t^2 \ln t \)

30. \( y = c_1 e^{2t} + c_2 e^{-t} - \frac{1}{2} (2t-1) e^{2t} \)

31. \( y = c_1 (1 + t) + c_2 e^t + \frac{1}{2} (t - 1) e^t \)

Section 3.7, page 262

1. \( u = 5 \cos(2t - \delta), \quad \delta = \arctan(4/3) \approx 0.9273 \)

2. \( u = 2 \cos(t - 2\pi/3) \)

3. \( u = 2\sqrt{3} \cos(3t - \delta), \quad \delta = -\arctan(1/2) \approx -0.4636 \)

4. \( u = \sqrt{13} \cos(\pi t - \delta), \quad \delta = \pi + \arctan(3/2) \approx 4.1244 \)

5. \( u = \frac{1}{2} \cos 8t \text{ ft}, \quad t \text{ in s}; \quad \omega = \frac{8}{9} \text{ rad/s}, \quad T = \pi/4 \text{ s}, \quad R = 1/4 \text{ ft} \)

6. \( u = \frac{2}{3} \sin 14t \text{ cm}, \quad t \text{ in s}; \quad t = \pi/14 \text{ s} \)

7. \( u = (1/4\sqrt{2}) \sin(\sqrt{2}t - \frac{1}{12} \cos(8\sqrt{2}t) \text{ ft}, \quad t \text{ in s}; \quad \omega = 8\sqrt{2} \text{ rad/s}, \quad T = \pi/4\sqrt{2} \text{ s}, \quad R = \sqrt{112} \approx 0.1954 \text{ ft}, \quad \delta = \pi - \arctan(3/\sqrt{2}) \approx 2.0113 \)

8. \( Q = 10^{-6} \cos 20000 \pi \text{ C}, \quad t \text{ in s} \)

9. \( u = e^{-\frac{1}{2}} \cos(4\sqrt{6}t - (5/\sqrt{6}) \sin(4\sqrt{6}t)) \text{ cm}, \quad t \text{ in s}; \quad \mu = 4\sqrt{6} \text{ rad/s}, \quad T_u = \pi/2\sqrt{6} \text{ s}, \quad T_{\phi}/T = 7/2\sqrt{6} \approx 1.4289, \quad \tau = 0.0404 \text{ s} \)

10. \( u = (1/8\sqrt{3}) e^{-\frac{1}{2}} \sin(2\sqrt{3}t) \text{ ft}, \quad t \text{ in s}; \quad t = \pi/2\sqrt{31} \text{ s}, \quad \tau = 1.5927 \text{ s} \)