5 Angles

5.1 The Angle Axioms and Basic Theorems

The following is a summary of Section 2.7 of Kay.

Axiom A-1 — Existence of Angle Measure: To every angle \( \angle A \) there corresponds a unique, real number \( \theta = m\angle A \), \( 0 < \theta < 180 \), called its measure.

Axiom A-2 — Angle Addition Postulate: If \( D \) lies in the interior of \( \angle ABC \), then \( m\angle ABC = m\angle ABD + m\angle DBC \), and conversely.

Axiom A-3 — Protractor Postulate: The set of rays having a common origin \( O \) and lying on one side of line \( \ell = \overrightarrow{OA} \), including ray \( \overrightarrow{OA} \), may be assigned to the real numbers \( \theta \) for which \( 0 \leq \theta < 180 \), called coordinates, in such a manner that

1. Each ray is assigned a unique coordinate \( \theta \).
2. Each coordinate \( \theta \) is assigned to a unique ray.
3. The coordinate of \( \overrightarrow{OA} \) is 0.
4. If rays \( \overrightarrow{OP} \) and \( \overrightarrow{OQ} \) have coordinates \( \theta \) and \( \phi \), respectively, then \( m\angle POQ = |\theta - \phi| \).

Definition: Suppose that \( \overrightarrow{OA} \), \( \overrightarrow{OB} \), and \( \overrightarrow{OC} \) are concurrent rays, all having the same endpoint \( O \). Then if these rays are distinct (no two are the same ray), and if \( m\angle AOB + m\angle BOC = m\angle AOC \), then \( \overrightarrow{OB} \) is said to lie between \( \overrightarrow{OA} \) and \( \overrightarrow{OC} \), and we write \( \overrightarrow{OA} \overrightarrow{OB} \overrightarrow{OC} \).
Notation: Write $A[a]$ if $a$ is the coordinate of a point $A$ under the Ruler Postulate. Write $\overrightarrow{OA}[a]$ if $a$ is the coordinate of a ray $\overrightarrow{OA}$ under the Protractor Postulate.

**Theorem 2.7.1:** If $A[a]$, $B[b]$, and $C[c]$ are three collinear points (and $\overrightarrow{OA}[a]$, $\overrightarrow{OB}[b]$, $\overrightarrow{OC}[c]$ three concurrent rays) with their coordinates, then $A-B-C$ ($\overrightarrow{OA}-\overrightarrow{OB}-\overrightarrow{OC}$) if and only if $a < b < c$ or $c < b < a$. (This is Theorem 1 of Kay, Section 2.7.)

**Corollary:** Suppose that four distinct collinear points are given with their coordinates: $A[a]$, $B[b]$, $C[c]$, $D[d]$. If $A-B-C$ and $A-C-D$, then $A-B-C-D$, and similarly for rays $\overrightarrow{OA}[a]$, $\overrightarrow{OB}[b]$, $\overrightarrow{OC}[c]$, $\overrightarrow{OD}[d]$.

**Lemma:** A segment, ray, or line is a convex set.

**Lemma:** If $A$ and $B$ are two distinct points, and $C \in \overrightarrow{AB}$, with $A \neq C$, then $\overrightarrow{AB} \subseteq \overrightarrow{AC}$.

**Theorem 2.7.2:** If $C \in \overrightarrow{AB}$ and $A \neq C$, then $\overrightarrow{AB} = \overrightarrow{AC}$. (This is Theorem 2 of Kay, Section 2.7.)

**Theorem 2.7.3 (Segment Construction Theorem):** If $\overrightarrow{AB}$ and $\overrightarrow{XY}$ are any two segments and $AB \neq XY$, then there is a unique point $C$ on ray $\overrightarrow{AB}$ such that $AC = XY$, with $A-C-B$ if $XY < AB$, or $A-B-C$ if $XY > AB$. (This is Theorem 3 of Kay, Section 2.7.)

36
**Definition:** A point $M$ on a segment $\overline{AB}$ is called a *midpoint* if it has the property that $AM = MB$. Such a midpoint is also said to *bisect* the segment, and any line, segment, or ray passing through that midpoint is also said to *bisect* the segment.

**Theorem 2.7.4 (Midpoint Construction Theorem):** The midpoint of any segment exists and is unique. (This is Theorem 4 of Kay, Section 2.7.)

**Theorem 2.7.5 (Segment Doubling Theorem):** There exists a unique point $C$ on ray $\overrightarrow{AB}$ such that $B$ is the midpoint of $\overline{AC}$. (This is Theorem 5 of Kay, Section 2.7.)

**Definition:** A ray $\overrightarrow{OM}$ such that $\overrightarrow{OA} - \overrightarrow{OM} - \overrightarrow{OB}$ is said to be an *angle bisector* of $\angle AOB$ if $m\angle AOM = m\angle OMB$. Any line or ray containing an angle bisector is said to *bisect* the angle.

**Theorem 2.7.3’ (Angle Construction Theorem):** If $\angle ABC$ and $\angle XYZ$ are any two nondegenerate angles and $m\angle ABC \neq m\angle XYZ$, then there exists a unique ray $\overrightarrow{BD}$ on the $C$-side of $\overrightarrow{AB}$ such that $m\angle XYZ = m\angle ABD$, and either $\overrightarrow{BA-\overrightarrow{BD}-\overrightarrow{BC}}$ if $m\angle XYZ < m\angle ABC$, or $\overrightarrow{BA-\overrightarrow{BC}-\overrightarrow{BD}}$ if $m\angle XYZ > m\angle ABC$. (This is Theorem 3’ of Kay, Section 2.7.)

**Theorem 2.7.4’ (Angle Bisection Theorem):** Every angle has a unique bisector. (This is Theorem 4’ of Kay, Section 2.7.)
Theorem 2.7.5’ (Angle Doubling Theorem): Given any angle $\angle ABC$ having measure $< 90$, there exists a ray $\overrightarrow{BD}$ such that $\overrightarrow{BC}$ is the bisector of $\angle ABD$. (This is Theorem 5’ of Kay, Section 2.7.)
5.2 More Theorems on Angles

This is a summary of Section 2.8 of Kay.

**Theorem 2.8.1 (Crossbar Theorem):** If $D$ is in the interior of $\angle BAC$, then ray $\overrightarrow{AD}$ meets segment $\overline{BC}$ at some interior point $E$. (This is Theorem 1 of Kay, Section 2.8.)

**Definition:** If $A-B-C$ then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called opposite rays.

**Lemma:** For every ray $\overrightarrow{PQ}$ there exists a unique opposite ray.

**Definition:** If the sides of one angle are opposite rays to the respective sides of another angle, the angles are said to form a vertical pair.

**Definition:** Two angles are said to form a linear pair iff they have one side in common and the other two sides are opposite rays. We call any two angles whose angle measures sum to 180 a supplementary pair, or more simply, supplementary, and two angles whose angle measures sum to 90 a complementary pair, or complementary.

**Theorem 2.8.2:** Two angles which are supplementary (or complementary) to the same angle have equal angle measures. (This is Theorem 2 of Kay, Section 2.8.)
**Axiom A-4:** A linear pair of angles is a supplementary pair.

**Theorem 2.8.3 (Vertical Pair Theorem):** Vertical angles have equal measures. (This is Theorem 3 of Kay, Section 2.8.)

**Definition:** If line \( \ell \) intersects another line \( m \) at some point \( A \) and forms a supplementary pair of angles at \( A \) having equal measures, then \( \ell \) is said to be **perpendicular** to \( m \), and we write \( \ell \perp m \).

**Definition:** An angle having measure 90 is called a **right angle**. Angles having measure less than 90 are **acute angles**, and those with measure greater than 90, **obtuse angles**.

**Theorem 2.8.4:** One line is perpendicular to another line iff the two lines form four right angles at their point of intersection. (This is Theorem 4 of Kay, Section 2.8.)

**Corollary:** Line \( \ell \) is perpendicular to line \( m \) iff \( \ell \) and \( m \) contains the sides of a right angle.

**Theorem 2.8.5 (Existence and Uniqueness of Perpendiculars):** Suppose that in some plane line \( m \) is given an an arbitrary point \( A \) on \( m \) is located. Then there exists a unique line \( \ell \) that is perpendicular to \( m \) at \( A \). (This is Theorem 5 of Kay, Section 2.8.)