

7 Quadrilaterals

This material is summarized from Section 3.7 of Kay.

**Definition:** If $A$, $B$, $C$, and $D$ are any four points lying in a plane such that no three of them are collinear, and if the points are so situated that no pair of open segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have any points in common, then the set $\triangle ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ is a quadrilateral, with vertices $A$, $B$, $C$, $D$; sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$; diagonals $\overline{AC}$, $\overline{BD}$, and angles $\angle DAB$, $\angle ABC$, $\angle BCD$, $\angle CDA$.

**Definition:** A quadrilateral is convex if its diagonals intersect.

**Lemma:** The diagonals of a convex quadrilateral intersect at an interior point on each diagonal.

**Lemma:** If $\triangle ABCD$ is a convex quadrilateral, then $D$ lies in the interior of $\triangle ABC$ (and similarly for the other vertices).

**Lemma:** If $A$, $B$, $C$, and $D$ are consecutive vertices of a convex quadrilateral, then $m \angle BAD = m \angle BAC + m \angle CAD$.

**Definition:** Two quadrilaterals $\triangle ABCD$ and $\triangle YZWX$ are congruent under the correspondence $ABC \leftrightarrow YZWX$ iff all pairs of corresponding sides and angles under the correspondence are congruent (i.e., CPCF). Such congruence will be denoted by $\triangle ABCD \cong \triangle YZWX$. 

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Theorem 3.7.1 (SASAS): Suppose that two convex quadrilaterals $\diamond ABCD$ and $\diamond XYZW$ satisfy the SASAS Hypothesis under the correspondence $ABCD \leftrightarrow XYZW$. That is, three consecutive sides and the two angles included by those sides of $\diamond ABCD$ are congruent, respectively, to the corresponding three consecutive sides and two included angles of $\diamond XYZW$. Then $\diamond ABCD \cong \diamond XYZW$. (This is Theorem 1 of Section 3.7 of Kay.)

There are other congruence theorems for, such as ASASA, SASAA, and SASSS.

What about ASAA?

**Definition:** A rectangle is a convex quadrilateral having four right angles.

**Definition:** Let $\overline{AB}$ be any line segment, and erect two perpendiculars at the endpoints $A$ and $B$. Mark off points $C$ and $D$ on these perpendiculars so that $C$ and $D$ lie on the same side of line $\overrightarrow{AB}$, and $BC = AD$. Join $C$ and $D$. The resulting quadrilateral is a Saccheri Quadrilateral. Side $\overline{AB}$ is called the base, $\overline{BC}$ and $\overline{AD}$ the legs, and side $\overline{CD}$ the summit. The angles at $C$ and $D$ are called the summit angles.

**Lemma:** A Saccheri Quadrilateral is convex.

**Theorem 3.7.2:** The summit angles of a Saccheri Quadrilateral are congruent. (This is Theorem 2 of Section 3.7 of Kay.)
Corollary:

1. The diagonals of a Saccheri Quadrilateral are congruent.
2. The line joining the midpoints of the base and the summit of a Saccheri Quadrilateral is the perpendicular bisector of both the base and summit.
3. If each of the summit angles of a Saccheri Quadrilateral is a right angle, the quadrilateral is a rectangle, and the summit is congruent to the base.
4. If the summit angles of a Saccheri Quadrilateral are acute, the summit has greater length than the base.

Theorem 3.7.3: Let \( \triangle ABC \) be any triangle. Let \( M \) and \( N \) be the midpoints of \( AB \) and \( AC \), respectively. Let \( BB' \) and \( CC' \) be perpendiculars to \( MN \), with \( B' \) and \( C' \) lying on \( MN \). Then \( \triangle BCC'B' \) is a Saccheri Quadrilateral with base \( B'C' \) and summit \( BC \). Moreover, the angle sum of \( \triangle ABC \) equals twice the measure of either summit angle of the quadrilateral, and \( MN = \frac{1}{2} B'C' \). (This is Theorem 3 of Section 3.7 of Kay.)

Corollary:

1. The summit angles of a Saccheri Quadrilateral are either acute or right.
2. The summit of a Saccheri Quadrilateral has length greater than or equal to that of the base.
3. The line joining the midpoints of two sides of a triangle has length less than or equal to one-half that of the third side.