8 Circles

8.1 Basic Results

This material is summarized from Section 3.8 of Kay.

**Definition:** A circle is the set of points in a plane which lie at a positive, fixed distance \( r \) from some fixed point \( O \). The number \( r \) is called the radius (as well as any line segment joining point \( O \) to any point on the circle), and the fixed point \( O \) is called the center of the circle. A point \( P \) is said to be interior to the circle, or an interior point, whenever \( OP < r \); if \( OP > r \), then \( P \) is said to be an exterior point.

Look at the diagram in the book to clarify the definitions of the following terms: diameter, radius, chord, secant (line), tangent (line) and point of contact or tangency, central angle, inscribed angle, semicircle, angle inscribed in a semicircle, arc, subtended or intercepted arc or chord of an angle.

**Lemma:**

1. The center of a circle is the midpoint of any diameter.

2. The perpendicular bisector of any chord of a circle passes through the center.

3. A line passing through the center of a circle and perpendicular to a chord bisects the chord.

4. Two congruent central angles subtend congruent chords, and conversely.

5. Two chords equidistant from the center of a circle have equal lengths, and conversely.
**Definition:** A *minor arc* is the intersection of the circle with a central angle and its interior, a *semicircle* is the intersection of the circle with a closed half-plane whose edge passes through $O$, and a *major arc* of a circle is the intersection of the circle and a central angle and its exterior (that is, the complement of a minor arc, plus endpoints). If the endpoints of an arc are $A$ and $B$, and $C$ is any other point of the arc (which must be used in order to uniquely identify the arc), then we define the *measure* $m\widehat{ACB}$ of the arc as follows:

1. Minor arc: $m\widehat{ACB} = m\angle AOB$.
2. Semicircle: $m\widehat{ACB} = 180$.
3. Major arc: $m\widehat{ACB} = 360 - m\angle AOB$.

Given a circle with center $O$ and ray $\overrightarrow{OP}$, let $H_1$ be one of the half-planes associated with $\overrightarrow{OP}$. Assign coordinates $0 \leq \theta < 180$ to $\overrightarrow{OP}$ and rays in this half-plane as before. Assign the coordinate 180 to the opposite ray of $\overrightarrow{OP}$. Assign coordinates $-180 < \theta < 0$ to the rays in the half-plane $H_2$ opposite to $H_1$, the negative of the coordinate that would have ordinarily been assigned with respect to $H_2$.

**Lemma:** For any arc $\widehat{ACB}$ on circle $O$, if $P'$ lies in the complementary arc of $\widehat{ACB}$ and $a > b$ are the coordinates of rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively, relative to the half-planes determined by line $\overrightarrow{PP'}$, then $m\widehat{ACB} = a - b$.

**Theorem 3.8.1:** Suppose arcs $A_1 = \widehat{ADC}$ and $A_2 = \widehat{CEB}$ are any two arcs of circle $O$ having just one point $C$ in common, and such that their union, $A_1 \cup A_2 = \widehat{ACB}$, is also an arc. Then $m(A_1 \cup A_2) = mA_1 + mA_2$. (This is Theorem 1 of Section 3.8 of Kay.)

67
**Theorem 3.8.2:** A line is tangent to a circle iff it is perpendicular to the radius at the point of contact. (This is Theorem 2 of Section 3.8 of Kay.)

**Theorem 3:** If a line $\ell$ passes through an interior point $A$ of a circle, it is a secant of the circle, intersecting the circle in precisely two points. (This is Theorem 3 of Section 3.8 of Kay.)
8.2 Circles on Spheres

Consider a circle of radius $r$ (as measured along the surface of a sphere) on a sphere of radius 1. Determine a formula for the circumference and the spherical area of the circle.