1 Basic Definitions

Definition 1.1 Let $x_1, \ldots, x_n \in \mathbb{R}^d$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ is

1. a linear combination of $x_1, \ldots, x_n$;
2. an affine combination of $x_1, \ldots, x_n$ if $\lambda_1 + \cdots + \lambda_n = 1$;
3. a nonnegative combination of $x_1, \ldots, x_n$ if $\lambda_1, \ldots, \lambda_n \geq 0$;
4. a convex combination of $x_1, \ldots, x_n$ if $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_1, \ldots, \lambda_n \geq 0$.

Remark 1.2 For $x_1, \ldots, x_n \in \mathbb{R}^d$, write

$$A = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

and

$$A' = \begin{bmatrix} x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{bmatrix}.$$  

Define also

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$  

Then the four cases in the above definition can be expressed, respectively, as

1. $x = A\lambda$.
2. $\begin{bmatrix} x \\ 1 \end{bmatrix} = A'\lambda$.
3. $x = A\lambda$, $\lambda \geq O$.
4. $\begin{bmatrix} x \\ 1 \end{bmatrix} = A'\lambda$, $\lambda \geq O$.

Definition 1.3 Let $x_1, \ldots, x_n \in \mathbb{R}^d$. Then $\{x_1, \ldots, x_n\}$ is

1. linearly dependent if $\exists \lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all zero, such that $\lambda_1 x_1 + \cdots + \lambda_n x_n = O$. Otherwise $\{x_1, \ldots, x_n\}$ is linearly independent.
2. affinely dependent if \( \exists \lambda_1, \ldots, \lambda_n \in \mathbb{R} \), not all zero, such that \( \lambda_1 x^1 + \cdots + \lambda_n x^n = 0 \) and \( \lambda_1 + \cdots + \lambda_n = 0 \). Otherwise \( \{x^1, \ldots, x^n\} \) is affinely independent.

**Theorem 1.4**

1. \( \{x^1, \ldots, x^n\} \subseteq \mathbb{R}^d \) is linearly dependent iff there exists one of the \( x^i \) that can be expressed as a linear combination of the others.

2. \( \{x^1, \ldots, x^n\} \subseteq \mathbb{R}^d \) is affinely dependent iff there exists one of the \( x^i \) that can be expressed as an affine combination of the others.

**Proof.** Exercise. \( \square \)

**Theorem 1.5** Let \( S = \{x^1, \ldots, x^n\} \subseteq \mathbb{R}^d \) and define \( A \) and \( A' \) as in Remark 1.2.

1. \( S \) is linearly independent iff \( \text{rank } A = n \), \( \text{dim} \{\lambda \in \mathbb{R}^n : A\lambda = 0\} = 0 \). In particular, if \( n > d \) then \( S \) is linearly dependent.

2. \( S \) is affinely independent iff \( \text{rank } A' = n \), \( \text{dim} \{\lambda \in \mathbb{R}^n : A'\lambda = 0\} = 0 \). In particular, if \( n > d + 1 \) then \( S \) is affinely dependent.

**Proof.** Exercise. \( \square \)

**Exercise 1.6** \( \{x^0, x^1, \ldots, x^n\} \subseteq \mathbb{R}^d \) is affinely independent iff \( \{x^1 - x^0, \ldots, x^n - x^0\} \) is linearly independent.

**Definition 1.7** Let \( S \subseteq \mathbb{R}^d \).

1. \( S \) is a linear set or subspace if \( S \neq \emptyset \) and \( \forall x, y \in S, \forall \lambda, \mu \in \mathbb{R}, \lambda x + \mu y \in S \). I.e., \( S \) is nonempty (in particular \( S \) contains \( O \)) and closed under all linear combinations of two elements.

2. \( S \) is an affine set, affine space, or flat if \( \forall x, y \in S, \forall \lambda, \mu \in \mathbb{R} \) such that \( \lambda + \mu = 1, \lambda x + \mu y \in S \). I.e., \( S \) is closed under all affine combinations of two elements.

3. \( S \) is a (convex) cone if \( S \neq \emptyset \) and \( \forall x, y \in S, \forall \text{nonnegative } \lambda, \mu \in \mathbb{R}, \lambda x + \mu y \in S \). I.e., \( S \) is nonempty (in particular \( S \) contains \( O \)) and is closed under all nonnegative combinations of two elements.

4. \( S \) is a convex set if \( \forall x, y \in S, \forall \text{nonnegative } \lambda, \mu \in \mathbb{R} \) such that \( \lambda + \mu = 1, \lambda x + \mu y \in S \). I.e., \( S \) is closed under all convex combinations of two elements.
Remark 1.8 We often (but not always) refer to elements of linear subspaces and convex cones as vectors and elements of affine and convex sets as points.

Theorem 1.9 Let \( \mathcal{L}, \mathcal{A}, \mathcal{K}, \mathcal{C} \) denote the collection of all linear subspaces, affine sets, cones, and convex sets, respectively, of \( \mathbb{R}^d \). Then \( \mathcal{A} \subseteq \mathcal{C} \) and \( \mathcal{K} \subseteq \mathcal{C} \), and these inclusions are strict. Further, \( \mathcal{A} \cap \mathcal{K} = \mathcal{L} \).

Proof. Exercise. \( \square \)

Exercise 1.10 Classify the following sets.

1. \( \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \).
2. \( \{ x \in \mathbb{R}^d : a^T x = \alpha \} \), where \( O \neq a \in \mathbb{R}^d \) and \( \alpha \in \mathbb{R} \). (Such a set is called a hyperplane.)
3. \( \{ x \in \mathbb{R}^d : Mx = O \} \) where \( M \) is an \( n \times d \) matrix.
4. \( \{ x \in \mathbb{R}^d : Mx \leq O \} \) where \( M \) is an \( n \times d \) matrix.
5. \( \{ x \in \mathbb{R}^d : Mx = b \} \) where \( M \) is an \( n \times d \) matrix and \( b \in \mathbb{R}^n \).
6. \( \{ x \in \mathbb{R}^d : Mx \leq b \} \) where \( M \) is an \( n \times d \) matrix and \( b \in \mathbb{R}^n \).

Theorem 1.11 Let \( S \subseteq \mathbb{R}^d \).

1. \( S \) is a linear subspace iff \( S \neq \emptyset \) and \( S \) is closed under linear combinations of finite numbers of vectors in \( S \).
2. \( S \) is an affine set iff \( S \) is closed under affine combinations of finite numbers of points in \( S \).
3. \( S \) is a cone iff \( S \neq \emptyset \) and \( S \) is closed under nonnegative combinations of finite numbers of vectors in \( S \).
4. \( S \) is a convex set iff \( S \) is closed under convex combinations of finite numbers of points in \( S \).

Proof. Exercise. \( \square \)

Theorem 1.12 Let \( S \) be a nonempty subset of \( \mathbb{R}^d \). Then \( S \) is an affine set iff there exists a linear subspace \( L \) of \( \mathbb{R}^d \) and a point \( y \) such that \( S = L + y \); i.e., \( S = \{ x + y : x \in L \} \). In this case \( L \) is unique, and \( y \) can be chosen to be any particular point of \( S \).
Theorem 1.13 Let $S \subseteq \mathbb{R}^d$.

1. $S$ is a linear subspace iff $S$ is a set of the form $\{x \in \mathbb{R}^d : Mx = O\}$ for some $n \times d$ matrix $M$.

2. $S$ is an affine set iff $S$ is a set of the form $\{x \in \mathbb{R}^d : Mx = b\}$ for some $n \times d$ matrix $M$ and $b \in \mathbb{R}^n$. If $S$ is nonempty, then in this case, $S = L + y$, where $y$ is any particular point in $S$ and $L = \{x \in \mathbb{R}^d : Mx = O\}$.

Proof. Exercise. □

Remark 1.14 Not all cones are of the form $\{x \in \mathbb{R}^d : Mx \leq O\}$. Those that are are called (convex) polyhedral cones. Similarly, not all convex sets are of the form $\{x \in \mathbb{R}^d : Mx \leq b\}$. Those that are are called (convex) polyhedra.

Proof. Exercise. □

Theorem 1.15

1. Every linear subspace is the intersection of a finite number of hyperplanes containing $O$.

2. Every affine set is the intersection of a finite number of hyperplanes.

Proof. Exercise. □

Theorem 1.16 The intersection of any collection of linear subspaces, affine sets, cones, convex sets is again a linear subspace, affine set, cone, convex set, respectively.

Proof. Exercise. □

Definition 1.17 Let $S \subseteq \mathbb{R}^d$.

1. The linear span of $S$, span $S$, is the intersection of all linear subspaces containing $S$.

2. The affine span of $S$, aff $S$, is the intersection of all affine sets containing $S$.

3. The cone of $S$, pos $S$, is the intersection of all cones containing $S$. 
4. The *convex hull* of $S$, conv $S$, is the intersection of all convex sets containing $S$.

**Definition 1.18** Let $S, T \subseteq \mathbb{R}^d$.

1. If $S = \text{span} T$, then we say $T$ (linearly) spans $S$.
2. If $S = \text{aff} T$, then we say $T$ affinely spans $S$.

**Theorem 1.19** Suppose $S$ is a nonempty subset of $\mathbb{R}^d$. Then span $S$, aff $S$, pos $S$, conv $S$ is the set of all linear, affine, nonnegative, convex combinations, respectively, of finite numbers of elements in $S$.

**Proof.** Exercise. □

**Definition 1.20** If $S \subseteq \mathbb{R}^d$ is a finite set, pos $S$ is called a *finite cone* and conv $S$ is called a *(convex) polytope.*

**Theorem 1.21** Let $S \subseteq \mathbb{R}^d$.

1. If $S \neq \{O\}$ is a linear subspace, then $\exists x^1, \ldots, x^n \in S$ such that every $x \in S$ can be expressed uniquely as a linear combination of $x^1, \ldots, x^n$. In this case, $n$ is the maximum size of a linearly independent subset of $S$ and the minimum size of a subset of $S$ that spans $S$ linearly.

2. If $S \neq \emptyset$ is an affine set, then $\exists x^1, \ldots, x^n \in S$ such that every $x \in S$ can be expressed uniquely as an affine combination of $x^1, \ldots, x^n$. In this case, $n$ is the maximum size of an affinely independent subset of $S$ and the minimum size of a subset of $S$ that spans $S$ affinely.

**Proof.** Exercise. □

**Definition 1.22** In the first case above, $\{x^1, \ldots, x^n\}$ is called a *basis* of $S$ and the *dimension* of $S$, dim $S$, equals $n$. If $S = \{O\}$, then dim $S = 0$. In the second case above, $\{x^1, \ldots, x^n\}$ is called an *affine basis* of $S$ and $S$ is the translate $L + y$ of some linear $(n - 1)$-dimensional subspace (verify this). We say that the dimension of $S$, dim $S$, equals $n - 1$. An affine set is 0-dimensional iff it is a set consisting of a single point. Affine sets of dimension 1 and 2 are called *lines* and *planes*, respectively.

**Exercise 1.23** Let $S$ be an affine subset of $\mathbb{R}^d$. Show that $S$ is a hyperplane iff dim $S = d - 1$. 

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Exercise 1.24 Let $S \subseteq \mathbf{R}^d$.

1. Show that a linear basis for $\text{span } S$ can be chosen from $S$ itself.

2. Show that an affine basis for $\text{aff } S$ can be chosen from $S$ itself.

Exercise 1.25 Suppose $M$ is an $n \times d$ matrix and $b \in \mathbf{R}^n$.

1. If $S = \{x \in \mathbf{R}^d : Mx = 0\}$, then $\dim S = d - \text{rank } M$.

2. If $\emptyset \neq S = \{x \in \mathbf{R}^d : Mx = b\}$, then $\dim S = d - \text{rank } M$.

Definition 1.26 Suppose $S$ is a nonempty subset of $\mathbf{R}^d$. The dimension of $S$, $\dim S$, is defined to be $\dim(\text{aff } S)$. (Note that the definitions of dimension agree if $S$ is an affine set.) The dimension of the empty set is defined to be $-1$. 

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