Today's Goal: We first define one-to-one functions, which in turn allows us to introduce the notion of inverse of a one-to-one function. These topics will be of particular importance when we study exponential and logarithmic functions.

Assignments: Homework (Sec. 3.7): # 1, 3, 7, 11, 17, 19, 21, 24, 31, 39, 47, 51, 52 (pp. 286-289).

Definition of a One-One Function:
A function \( f \) with domain \( A \) is called a one-to-one function if no two elements of \( A \) have the same image, that is,
\[
f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.
\]
An equivalent way of writing the above condition is:
If \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test:
A function is one-to-one \( \iff \) no horizontal line intersects its graph more than once.

Example 1: Show that the function \( f(x) = 5 - 2x \) is one-to-one.
\[
y = 5 - 2x \text{ is a line with slope } -2, \quad \text{so } y = f(x) \text{ passes the horizontal line test.}
\]

(Also if \( 5 - 2x = 5 - 2x' \), then \( -2x = -2x' \) and \( x = x' \), so \( f \) is one-to-one.)

Example 2:
Graph the function \( f(x) = (x - 2)^2 - 3 \). The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)

\( f(x) \) is \( x^2 \) shifted down three units and to the right two units:

Every point other than the vertex lies on a horizontal line which intersects the graph at two different places, so \( f \) is not one-to-one.

If the domain is restricted to \([2, \infty) \) or \((-\infty, 2] \), the function becomes one-to-one (because half of the parabola is chopped off).

Note: The domain can always be restricted to a single value so that no two elements will have the same image.
The Inverse of a Function: One-to-one functions are precisely those for which one can define a (unique) inverse function according to the following definition.

**Definition of the Inverse of a Function:**

Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by

\[
 f^{-1}(y) = x \iff f(x) = y,
\]

for any \( y \in B \).

If \( f \) takes \( x \) to \( y \), then \( f^{-1} \) takes \( y \) back to \( x \).

I.e., \( f^{-1} \) undoes what \( f \) does.

**NOTE:**

\( f^{-1} \) does NOT mean \( \frac{1}{f} \).

**Example 3:** Suppose \( f(x) \) is a one-to-one function.

If \( f(2) = 7, \ f(3) = -1, \ f(5) = 18, \ f^{-1}(2) = 6 \) find:

\[
 f^{-1}(7) = 2 \quad f(6) = 2 \\
 f^{-1}(-1) = 3 \\
 f(f^{-1}(18)) = f(5) = 18
\]

If \( g(x) = 9 - 3x \), then \( g^{-1}(3) = \)

\[
 9 - 3x = 3 \Rightarrow -3x = -6 \Rightarrow x = 2
\]

**Property of Inverse Functions:**

Let \( f(x) \) be a one-to-one function with domain \( A \) and range \( B \).

The inverse function \( f^{-1}(x) \) satisfies the following "cancellation" properties:

\[
 f^{-1}(f(x)) = x \quad \text{for every } x \in A \\
 f(f^{-1}(x)) = x \quad \text{for every } x \in B
\]

Conversely, any function \( f^{-1}(x) \) satisfying the above conditions is the inverse of \( f(x) \).

**Example 4:** Show that the functions \( f(x) = x^5 \) and \( g(x) = x^{1/5} \) are inverses of each other.

\[
 f(g(x)) = f(x^{1/5}) = (x^{1/5})^5 = x^1 = x
\]

and

\[
 g(f(x)) = g(x^5) = (x^5)^{1/5} = x^1 = x
\]

**Example 5:** Show that the functions \( f(x) = \frac{1 + 3x}{5 - 2x} \) and \( g(x) = \frac{5x - 1}{2x + 3} \) are inverses of each other.

\[
 f(g(x)) = f\left(\frac{5x - 1}{2x + 3}\right) = \frac{1 + 3\left(\frac{5x - 1}{2x + 3}\right)}{5 - 2\left(\frac{5x - 1}{2x + 3}\right)} = \frac{(2x+3)+3(5x-1)}{5(2x+3)-2(5x-1)}
\]

\[
 = \frac{2x+3+15x-2}{10x+15-10x+2} = \frac{17x}{17} = x
\]

\[
 g(f(x)) = g\left(\frac{1 + 3x}{5 - 2x}\right) = \frac{5\left(\frac{1 + 3x}{5 - 2x}\right)-1}{2\left(\frac{1 + 3x}{5 - 2x}\right)+3} = \frac{5(1+3x)-1}{2(1+3x)+3(5-2x)}
\]

\[
 = \frac{5+15x-5+6x}{2+6x+15-6x} = \frac{17x}{17} = x
\]

**How to find the Inverse of a One-to-One Function:**

1. Write \( y = f(x) \).
2. Solve this equation for \( x \) in terms of \( y \) (if possible).
3. Interchange \( x \) and \( y \). The resulting equation is \( y = f^{-1}(x) \).

**Example 6:** Find the inverse of \( y = 4x - 7 \).

\[
 y = 4x - 7 \quad \rightarrow \quad y = \frac{x + 7}{4}
\]

\[
 y + 7 = 4x \\
 \frac{y + 7}{4} = x
\]
Example 7: Find the inverse of \( y = \frac{1}{x+2} \).

\[
\begin{align*}
y & = \frac{1}{x+2} \\
(y+2) y & = 1 \\
x+2 & = \frac{1}{y} \\
x & = \frac{1}{y} - 2
\end{align*}
\]

Example 8: Find the inverse of \( y = \frac{2-x}{x+2} \).

\[
\begin{align*}
y & = \frac{2-x}{x+2} \\
(x+2) y & = 2-x \\
x y + 2y & = 2-x \\
xy + x & = 2-2y \\
xy + 1 & = 2-2y \\
x & = \frac{2-2y}{y+1}
\end{align*}
\]

Graph of the Inverse Function: The principle of interchanging \( x \) and \( y \) to find the inverse function also gives us a method for obtaining the graph of \( f^{-1} \) from the graph of \( f \). The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) in the line \( y = x \).

The picture on the right hand side shows the graphs of:
\[
f(x) = \sqrt{x+4} \quad \text{and} \quad f^{-1}(x) = x^2 - 4, \ x \geq 0.
\]

Example 9: Find the inverse of the function \( f(x) = 1 + \sqrt{1+x} \).

Find the domain and range of \( f \) and \( f^{-1} \). Graph \( f \) and \( f^{-1} \) on the same cartesian plane.

\( f(x) = 1 + \sqrt{1+x} \) has a domain of \([-1, \infty)\) because \( 1+x \geq 0 \Rightarrow x \geq -1 \).

\( f(x) = 1 + \sqrt{1+x} \) has a range of \([1, \infty)\) because \( \sqrt{1+x} \geq 0 \) so \( 1 + \sqrt{1+x} \geq 1 \).

Thus, the domain of \( f^{-1} \) is \([1, \infty)\), and the range of \( f^{-1} \) is \([-1, \infty)\).