Today’s Goal: So far we have been studying polynomial functions graphically. We now begin to study polynomials algebraically. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

Assignments: Homework (Sec. 4.2): # 1, 3, 5, 11, 13, 19, 22, 27, 31, 36, 43, 53 (pp. 331-332).

Given the integers 23 and 5 we can ‘divide’ one by the other. We obtain: \( \frac{23}{5} = 4 + \frac{3}{5} \) or \( 23 = 4 \cdot 5 + 3 \).

In general, if \( a \) and \( b \) are non-zero integers, then there exist unique integers \( q \) and \( r \) such that

\[ a = q \cdot b + r \quad \text{and} \quad 0 \leq r < |b|, \]

where \( q \) is the quotient and \( r \) the remainder. This is the usual ‘long division’ familiar from elementary arithmetic.

**Example 1:** Divide 63 by 12.

\[ \frac{63}{12} = 5 + \frac{3}{12} \quad \Rightarrow \quad 63 = 5 \cdot 12 + 3 \]

**[Long Division of Polynomials]** Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials.*

**Division Algorithm:** If \( P(x) \) and \( D(x) \) are polynomials, with \( D(x) \neq 0 \), then there exist unique polynomials \( Q(x) \) and \( R(x) \), where \( R(x) \) is either 0 or of degree strictly less than the degree of \( D(x) \), such that

\[ P(x) = Q(x) \cdot D(x) + R(x) \]

The polynomials \( P(x) \) and \( D(x) \) are called the *dividend* and *divisor*, respectively; \( Q(x) \) is the *quotient* and \( R(x) \) is the *remainder*.

**Example 2:** Divide the polynomial

\[ P(x) = 2x^2 - x - 4 \quad \text{by} \quad D(x) = x - 3. \]

\[ \begin{array}{c}
2x^2 - x - 4 \\
\sqrt{x - 3} \\
2x^2 - x - 4 \\
- \frac{2x}{2x^2 - x - 4} \\
\frac{-6x}{+ 5x - 4} \\
\frac{3(5x - 15)}{+ 11} \\
\end{array} \]

\[ 2x^2 - x - 4 = (2x+5)(x-3) + 11 \]

\[ \checkmark: \quad (2x+5)(x-3) + 11 = 2x^2 - x - 4 \]

\[ \checkmark: \quad (2x+5)(x-3) + 11 = 2x^2 - x - 4 \]

(Complete the above table and check your work!)

**Example 3:** Divide the polynomial

\[ P(x) = x^4 - x^3 + 4x + 2 \quad \text{by} \quad D(x) = x^2 + 3. \]

\[ \begin{array}{c}
x^4 - x^3 + 4x + 2 \\
\sqrt{x^2 + 3} \\
x^4 - x^3 + 4x + 2 \\
- (x^2 + 3x^2) \\
- (x^2 - 3x + 2) \\
- (-3x^2 - 7x + 9) \\
\end{array} \]

\[ x^4 - x^3 + 4x + 2 = (x^2-x-3)(x^2+3) + 7x+11 \]

\[ \checkmark: \quad (x^2-x-3)(x^2+3) + 7x+11 = x^4 - x^3 + 4x + 2 \]

\[ \checkmark: \quad (x^2-x-3)(x^2+3) + 7x+11 = x^4 - x^3 + 4x + 2 \]
Example 4:
Find the quotient $Q(x)$ and the remainder $R(x)$ when $f(x) = 3x^3 + 2x^2 - x + 3$ is divided by $g(x) = x - 4$.

\[
\begin{array}{c}
Q(x) = \frac{3x^2 + 14x + 55}{3x^2 + 14x + 55} \\
= \frac{3x^3 - 12x^2 + 14x^2 - x + 3}{-(3x^3 - 12x^2)}
\end{array}
\]

\[
\begin{array}{c}
\text{Quotient:} \quad \frac{3x^3 + 14x^2 + 55}{x - 4} \\
\text{Remainder:} \quad 223
\end{array}
\]

\[
\begin{array}{c}
3x^3 + 14x^2 + 55x - 12x^2 - 56x - 220 \\
= 3x^3 + 2x^2 - x + 3 \quad \checkmark
\end{array}
\]

Example 5:
Find the quotient $Q(x)$ and the remainder $R(x)$ when $f(x) = x^5 - 4x^4 + x$ is divided by $g(x) = x + 3$.

\[
\begin{array}{c}
Q(x) = \frac{x^4 - 3x^2 + 5x^2 - 15x + 46}{x^5 + 3x^4 - 4x^3 + 0x^2 + x + 0} \\
- (x^4 + 3x^3)
\end{array}
\]

\[
\begin{array}{c}
R(x) = \frac{5x^3 + 0x^2 - 7}{-(5x^3 + 15x^2)} \\
- (15x^2 - 45x)
\end{array}
\]

\[
\begin{array}{c}
\text{The Remainder and Factor Theorems:} \\
\text{Next, we see how synthetic division can be used to evaluate polynomials easily.}
\end{array}
\]

- **Remainder Theorem:**
  If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

- **Proof:** If the divisor $D(x)$ is of the form $x - c$, then the remainder MUST be a constant $R$. Thus
  \[ P(x) = Q(x) \cdot (x - c) + R. \]

  Setting $x = c$ in the above equation gives that $P(c) = Q(c) \cdot 0 + R = R$. Thus
  \[ P(x) = Q(x) \cdot (x - c) + P(c). \]

  From the boxed equation we obtain our next theorem, which says that the zeros of a polynomial correspond to the linear factors of the polynomial.

- **Factor Theorem:**
  The number $c$ is a zero of $P(x)$ if and only if $x - c$ is a factor of $P(x)$;
  that is, $P(x) = Q(x) \cdot (x - c)$ for some polynomial $Q(x)$.

Example 6: Let $P(x) = x^3 + 2x^2 - 7$.

(a) Find the quotient and the remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$.

\[
\begin{array}{c}
\text{(a) } x+2 \quad \frac{x^3 + 2x^2 + 0x - 7}{x^3 + 2x^2} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
Q(x) = \frac{6}{x + 2} \\
P(-2) = -7
\end{array}
\]

So
\[
\begin{array}{c}
x^3 + 2x^2 - 7 = x^2 (x + 2) - 7
\end{array}
\]

(b) By the Remainder Theorem, $P(-2) = -7$.

\[
\begin{array}{c}
P(-2) = (-2)^3 + 2(-2)^2 - 7 = -8 + 8 - 7 = -7 \quad \checkmark
\end{array}
\]
Example 7: Use the Factor Theorem to determine whether \(x + 2\) is a factor of \(f(x) = 3x^6 + 2x^3 - 176\).

According to the Factor Theorem, \(x + 2\) is a factor of \(f(x)\) if and only if \(-2\) is a zero of \(f(x)\), i.e., \(f(-2) = 0\).

\[
f(-2) = 3(-2)^6 + 2(-2)^3 - 176 = 3(64) + 2(-8) - 176 = 192 - 16 - 176 = 192 - 192 = 0
\]

So \(x + 2\) is a factor of \(f(x) = 3x^6 + 2x^3 - 176\).

Example 8: Find a polynomial of degree 3 that has zeros 1, \(-2\), and 3, and in which the coefficient of \(x^2\) is 3.

If your polynomial has zeros 1, \(-2\), and 3, it has factors \((x-1), (x+2)\) and \((x-3)\). Notice \((x-1)(x+2)(x-3) = (x^2 + x - 2)(x-3) = x^3 + x^2 - 2x - 3x^2 - 6x + 6 = x^3 - 2x^2 - 9x + 6\).

Now if we want the coefficient of \(x^2\) to be 3, we need to multiply \((-2)(-2) = 4\). \((-\frac{3}{2})(x-1)(x+2)(x-3) = -\frac{3}{2}(x^3 - 2x^2 - 9x + 6)\)

Example 9: Let \(P(x) = 2x^3 + 3x^2 - 17x - 30\).

- Is 3 a zero of \(P(x)\)? What does this tell you about the factors of \(P(x)\)?
  - What does it tell you about the graph of \(y = P(x)\)?

- Is 2 a zero of \(P(x)\)? What does this tell you about the factors of \(P(x)\)?
  - What does it tell you about the graph of \(y = P(x)\)?

- Is 3 a zero of \(P(x)\)? \(P(3) = 2(3)^3 + 3(3)^2 - 17(3) - 30 = 54 + 27 - 51 - 30 = 0\)
  - Yes, 3 is a zero of \(P(x)\). This means \((x-3)\) is a factor of \(P(x)\) and the graph of \(y = P(x)\) touches or crosses the \(x\)-axis at \(x = 3\).

- Is 2 a zero of \(P(x)\)? \(P(2) = 2(2)^3 + 3(2)^2 - 17(2) - 30 = 16 + 12 - 34 - 30 = 28 - 46 = -18\) \(\neq 0\).
  - No, 2 is not a zero of \(P(x)\).
  - So \((x-2)\) is not a factor of \(P(x)\) and the graph of \(y = P(x)\) does not touch or cross the \(x\)-axis at \(x = 2\).

Example 10: The graph of a polynomial has \(x\)-intercepts at \((2,0)\) and \((-5,0)\). What does this tell you about the polynomial?

This means \(x = 2\) and \(x = -5\) are zeros of this polynomial, which, in turn, means \((x-2)\) and \((x+5)\) are factors of the polynomial.

\[
\text{Remember } x - (-5) = x + 5.
\]