A quadratic equation is an equation of the form:

$$ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are real numbers with $a \neq 0$.

**Zero-Product Property:** For any $A, B \in \mathbb{R}$:

$$AB = 0 \quad \text{if and only if} \quad A = 0 \text{ or } B = 0.$$  

**Example 1:** Solve the following equations by factoring:

- $x^2 - 7x + 12 = 0$
  
  $$(x - 3)(x - 4) = 0$$
  
  $x - 3 = 0$, $x = 3$
  
  $x - 4 = 0$, $x = 4$

- $2x^2 = x + 3$
  
  $$2x^2 - x - 3 = 0$$
  
  $$(2x - 3)(x + 1) = 0$$
  
  $2x - 3 = 0$, $x = \frac{3}{2}$
  
  $x + 1 = 0$, $x = -1$

**Solving Quadratic Equations by Completing the Square:**

If a quadratic equation is of the form

$$(x \pm \alpha)^2 = \beta,$$

we can solve it by taking the square root of each side. So, if a quadratic equation does not factor readily ... we solve it by completing the square!

**Completing the Square:** To make a perfect square out of $x^2 + bx$, add the square of half the coefficient of $x$, that is $(b/2)^2$. Thus:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

**Example 2:** Solve each equation by completing the square:

- $x^2 + 4x - 6 = 0$
  
  $$x^2 + 4x = 6$$
  
  $$(x + 2)^2 = 10$$
  
  $x + 2 = \pm \sqrt{10}$
  
  $x = -2 \pm \sqrt{10}$

- $3x^2 - 6x - 1 = 0$
  
  $$3x^2 - 6x = \frac{1}{3}$$
  
  $$x^2 - 2x = \frac{1}{3}$$
  
  $$(x - 1)^2 = \frac{4}{3}$$
  
  $x - 1 = \pm \frac{2}{\sqrt{3}}$
  
  $x = 1 \pm \frac{2}{\sqrt{3}}$
We can use the technique of completing the square to derive a formula for the general quadratic equation:

\[ ax^2 + bx + c = 0 \]

(where \( a \neq 0 \)). We obtain the following:

**The Quadratic Formula:**

The roots \( x_1 \) and \( x_2 \) of the quadratic equation \( ax^2 + bx + c = 0 \), where \( a \neq 0 \), are:

\[
x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

**Example 3:** Find all solutions of each equation:

- \( 3x^2 + 7x + 4 = 0 \)
  
  \[
x_1, x_2 = \frac{-7 \pm \sqrt{7^2 - 4(3)(4)}}{2(3)}
  \]
  
  \[
x_1 = \frac{-7 + 1}{6} = -1, \quad x_2 = \frac{-7 - 1}{6} = -\frac{8}{6}
  \]
  
  \[ x_1 = -1 - \frac{1}{6}, \quad x_2 = -\frac{4}{3} \]

- \( 9 + \frac{3}{x} - \frac{2}{x^2} = 0 \)
  
  \[
x_1, x_2 = \frac{1 \pm \sqrt{1^2 - 4(-1)(1)}}{2(-1)}
  \]
  
  \[
x_1, x_2 = \frac{1 \pm \sqrt{-15}}{2} \Rightarrow \text{No Real Roots}
  \]

**The Discriminant:**

The discriminant \( D \) of the quadratic equation \( ax^2 + bx + c = 0 \), where \( a \neq 0 \), is:

\[
D = b^2 - 4ac
\]

1. If \( D > 0 \) the eq. has 2 distinct real roots.
2. If \( D = 0 \) the eq. has exactly 1 real root.
3. If \( D < 0 \) the eq. has no real roots.

**Example 4:** Use the discriminant to determine how many real roots each equation has. Do not solve the equation.

- \( 3x^2 - 5x + 1 = 0 \)
  
  \[
D = (5)^2 - 4(3)(1)
  \]
  
  \[ D = 25 - 12 = 13 > 0 \Rightarrow 2 \text{ real roots} \]

- \( x^2 = 6x - 10 \)
  
  \[
D = (6)^2 - 4(1)(10)
  \]
  
  \[ D = 36 - 40 \Rightarrow -4 < 0 \text{ No Real Roots} \]

**Proof:**

\[
a x^2 + b x + c = 0 \]

\[
s x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Example 5:
Find all values of \( k \) that ensure that the equation
\[
kx^2 + 36x + k = 0
\]
has exactly one root (solution).

\[
D = 0 \implies 1 \text{ REAL SOLUTION}
\]
\[
0 = 36^2 - 4(k)(k)
\]
\[
0 = 36^2 - 4k^2
\]
\[
4k^2 = 36^2
\]
\[
k = \pm \frac{36}{2} = \pm 18
\]

**Modeling with Quadratic Equations:**
The principles discussed in Activity 2 for setting up equations as models are useful here as well.

**Example 6 (Dimension of a Lot):** A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite one is 174 ft long. What are the dimensions of the parcel?

\[
\begin{align*}
\text{Diagonal: } & \quad 174^2 = x^2 + (x + 6)^2 \\
& \implies 174^2 = x^2 + x^2 + 12x + 36
\end{align*}
\]

\[
\begin{align*}
2x^2 + 12x - 30240 &= 0 \\
2(x^2 + 6x) - 30240 &= 0
\end{align*}
\]

\[
x_1 = -12 + 12\sqrt{422} = 120
\]

\[
x_2 = -12 - 12\sqrt{422} = -126
\]

**Example 7 (Falling-Body Problem):**
An object is thrown straight upward at an initial speed of 400 ft/s. From Physics, it is known that, after \( t \) seconds, it reaches a height of \( h \) feet given by the formula:
\[
h = -16t^2 + 400t.
\]

(a) When does the object fall back to ground level?

\[
h = 0
\]

\[
0 = -16t^2 + 400t
\]

\[
x_1, x_2 = \frac{-400 \pm \sqrt{400^2 - 4(-16)(0)}}{2(-16)}
\]

\[
\begin{align*}
x_1, x_2 &= \frac{-400 + 400}{32} = 0 \\
x_2 &= \frac{-400 - 400}{-32} = 12.5
\end{align*}
\]

(b) When does it reach a height of 1,600 ft?

\[
1600 = -16t^2 + 400t
\]

\[
x_3, x_4 = \frac{-400 \pm \sqrt{400^2 - 4(-16)(1600)}}{2(-16)}
\]

\[
\begin{align*}
x_3, x_4 &= \frac{-400 \pm 1200}{-32} = -5, 20
\end{align*}
\]

(c) When does it reach a height of 1 mi? (1 mi = 5,280 ft)

\[
5280 = -16t^2 + 400t
\]

\[
0 = -16t^2 + 400t - 5280
\]

\[
x_5, x_6 = \frac{-400 \pm \sqrt{400^2 - 4(-16)(-5280)}}{2(-16)}
\]

\[
\begin{align*}
x_5, x_6 &= \frac{-400 \pm 173420}{-32} = -400, 520
\end{align*}
\]

(d) How high is the highest point the object reaches?

\[
\text{OBJECT reaches the highest point twice.}
\]

\[
\text{So Discriminant = 0}
\]

\[
0 = -16t^2 + 400t - h
\]

\[
D = 400^2 - 4(-16)(-h) = 0
\]

\[
h = 2500
\]