Today's Goal: We learn how to solve quadratic (or second-degree) equations.

Assignments: Homework (Sec. 1.3): # 1, 8, 13, 19, 23, 29, 38, 43, 71, 75, 79, 89, 91 (pp. 105-108). Reading for next lecture: Read Sec. 1.5 (pp. 115-121).

A quadratic equation is an equation of the form:

$$ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are real numbers with $a \neq 0$.

Zero-Product Property: For any $A, B \in \mathbb{R}$:

$$AB = 0 \quad \text{if and only if} \quad A = 0 \text{ or } B = 0.$$

- **Solving Quadratic Equations by Completing the Square:**

If a quadratic equation is of the form

$$(x \pm a)^2 = \beta,$$

we can solve it by taking the square root of each side. So, if a quadratic equation does not factor readily ... we solve it by completing the square!

**Completing the Square:** To make a perfect square out of $x^2 + bx$, add the square of half the coefficient of $x$, that is $(b/2)^2$. Thus:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

**Example 2:** Solve each equation by completing the square:

- $x^2 + 4x - 6 = 0$
  $$x^2 + 4x + 4 = 6 + 4$$
  $$\Rightarrow \quad (x + 2)^2 = 10$$
  $$\Rightarrow \quad x = -2 \pm \sqrt{10}$$

- $3x^2 - 6x - 1 = 0$
  $$3(x^2 - 2x) = 1$$
  $$\Rightarrow \quad 3(x^2 - 2x + 1) = 1 + 3$$
  $$\Rightarrow \quad 3(x - 1)^2 = 4$$
  $$\Rightarrow \quad (x - 1)^2 = \frac{4}{3}$$
  $$\Rightarrow \quad x = 1 \pm \frac{2}{3}$$

**Geometric Interpretation of Completing the Square:**

This interpretation goes back to the Babylonian scribes, who fully used the "cut-and-paste" geometry developed by the ancient surveyors (ca. 1700 BC). Here, $x$ and $b$ are positive as they represent lengths:

$$x + \frac{b}{2}$$

where $\square$ is a square of side $b/2$; thus its area is $(b/2)^2$.

**The Quadratic Formula:**

The roots $x_1$ and $x_2$ of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Example 3: Find all solutions of each equation:

- $3x^2 + 7x + 4 = 0$
  \[ x = \frac{-7 \pm \sqrt{49-4 \cdot 3 \cdot 4}}{6} = \frac{-7 \pm \sqrt{49-48}}{6} = \frac{-7 \pm 1}{6} \]
  \[ x = \frac{-7 + 1}{6} = -1 \quad \text{or} \quad x = \frac{-7 - 1}{6} = -\frac{4}{3} \]

- $x = 1 - \frac{4}{x}$
  \[ x \left( x = 1 - \frac{4}{x} \right) \Rightarrow x^2 - x + 4 = 0 \]
  \[ x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 4}}{2} = \frac{1 \pm \sqrt{15}}{2} \]
  \[ x^2 = x - 4 \]
  \[ \text{no real solutions} \]

The Discriminant:

The discriminant $D$ of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, is:

$D = b^2 - 4ac$

1. If $D > 0$ the eq. has 2 distinct real roots.
2. If $D = 0$ the eq. has exactly 1 real root.
3. If $D < 0$ the eq. has no real roots.

Example 4: Use the discriminant to determine how many real roots each equation has. Do not solve the equation.

- $2t^2 + 5t + 3 = 0$
  \[ (2t + 3)(t + 1) = 0 \]
  \[ t = -\frac{3}{2} \quad \text{or} \quad t = -1 \]

- $9 + \frac{3}{x - x^2} = 0$
  \[ 9x^2 + 3x - 2 = 0 \]
  \[ (3x + 2)(3x - 1) = 0 \]
  \[ x = -\frac{2}{3} \quad \text{or} \quad x = \frac{1}{3} \]

Example 5: Find all values of $k$ that ensure that the equation

$kx^2 + 36x + k = 0$

has exactly one root (solution).

$D = 36^2 - 4k^2 = 0$

$36^2 = 4k^2$

$k = \frac{\pm 36}{2} = \pm 18$

Example 6 (Dimension of a Lot):

A parcel of land is 6 ft longer than it is wide. Each diagonal from one corner to the opposite one is 174 ft long. What are the dimensions of the parcel?

Example 7 (Falling-Body Problem):

An object is thrown straight upward at an initial speed of 400 ft/s. From Physics, it is known that, after $t$ seconds, it reaches a height of $h$ feet given by the formula:

$h = -16t^2 + 400t$.

(a) When does the object fall back to ground level?

(b) When does it reach a height of 1,600 ft?

(c) When does it reach a height of 1 mi? (1 mi = 5,280 ft)

(d) How high is the highest point the object reaches?

$h = -16t^2 + 400t = -16(t - 25)^2 + 2500$