MA 113 Calculus I  Spring 2014
Exam 3  Tuesday, 15 April 2014

Name: 

Section: 

Last 4 digits of student ID #: 

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

- Select your answer by placing an X in the appropriate square of the multiple choice answer box on the front page of the exam.
- Carefully check your answers. No credit will be given for answers other than those indicated on the multiple choice answer box.

On the free response problems:

- Clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may not receive credit),
- Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question.
1. Let \( f(x) = x^3 - 2x^2 \). Find the largest open interval on which \( f \) is decreasing.

(A) (0, 4/3)  
(B) (0, 2)  
(C) \((-\infty, 2/3)\)  
(D) \((2/3, \infty)\)  
(E) \((-2, 0)\)

\[
\begin{align*}
\text{f decre \( \iff \) } f' &< 0. \\
f'(x) &= 3x^2 - 4x = x(3x - 4) \\
\text{2 pts. } f'(c) = 0 & : x = 0 \text{ & } x = \frac{4}{3}.
\end{align*}
\]

\[
\begin{array}{c}
\text{sign}(f') \\
\text{pos} & \quad 0 < x < \frac{4}{3} \\
\text{neg} & \quad \frac{4}{3} < x
\end{array}
\]

so \( x \in (0, \frac{4}{3}) \) \( \text{\(\bigcirc\)} \)

2. If \( f'(1) = 0 \) and \( f''(1) > 0 \), then which of the following is false

(A) \( f \) has a critical point at 1.  
(B) \( f \) has a local minimum at 1  
(C) \( f \) has a local maximum at 1  
(D) \( f \) is differentiable at 1  
(E) \( f \) is continuous at 1

(c) since \( f'' > 0 \) means \( f \) is concave up near 1 so a local min.
3. Give the linear approximation to \( f(x) = \sqrt{3x + 3} \) at \( x = 2 \).

(A) \( L(x) = \frac{1}{2} x + 2 \)

(B) \( L(x) = \frac{1}{2} x \)

(C) \( L(x) = \frac{1}{2} x + \frac{1}{2} \)

(D) \( L(x) = \frac{3}{2} x \)

(E) \( L(x) = \frac{3}{2} x - \frac{5}{2} \)

\[
\begin{align*}
f'(x) &= \frac{1}{2} (3x + 3)^{-\frac{1}{2}} \\
f'(2) &= \frac{1}{2} \\
f(2) &= 3 \\
y - f(2) &= \frac{1}{2} \sqrt{x - 2} \\
y'(x) &= \frac{1}{2} (x - 2) + f(2) \\
&= \frac{1}{2} x - 1 + 3 = \frac{1}{2} x + 2. \quad \text{(A)}
\end{align*}
\]

4. Let \( f(x) = \sin(2x) \). On which subintervals of \([0, \pi]\) is \( f \) concave down?

(A) \((0, \pi/4), (3\pi/4, \pi)\)

(B) \((\pi/4, 3\pi/4)\)

(C) \((0, \pi/2)\)

(D) \((\pi/2, \pi)\)

(E) \((0, \pi)\)

\[
\begin{align*}
f'(x) &= 2 \cos(2x) \\
f''(x) &= -4 \sin(2x) \\
f''(c) &= 0 \text{ for } c \in (0, \pi) \text{ if } x = \frac{\pi}{2}
\end{align*}
\]

You can also sketch \( \sin(2x) \):

\[
\text{neg} \quad \text{pos} \\
\begin{array}{c|c|c|c}
0 & \frac{\pi}{2} & \pi \\
\downarrow & \uparrow & \downarrow
\end{array}
\]

so \( f'' < 0 \) for concave down

\( \Rightarrow (0, \frac{\pi}{2}) \quad \text{(C)} \)
5. Let \( f(x) = x^3 + 3x \). Which of the following statements is true?

(A) has no local extrema
(B) \( f \) has a local maximum at \(-1\)
(C) \( f \) has a local maximum at \(1\)
(D) \( f \) has a local minimum at \(1\)
(E) \( f \) has a local minimum at \(-1\)

\[
f'(x) = 3x^2 + 3 = 3(x^2 + 1) \quad \text{so no critical pts.}
\]

(f is a polynomial)

Whence no local extrema  \(\Box\)

6. Let \( f(x) = \frac{(2x+1)(x-2)(1-x)}{x^3} \). and find \( \lim_{x \to \infty} f(x) \).

(A) 0
(B) 2
(C) \(-2\)
(D) \(\infty\)
(E) \(-\infty\)

\[
f(x) = \frac{x^3 \left( 2 + \frac{1}{x} \right) \left( 1 - \frac{2}{x} \right) \left( \frac{1}{x} - 1 \right)}{x^3}
\]

\[
= -\left( 2 + \frac{1}{x} \right) \left( 1 - \frac{2}{x} \right) \left( 1 - \frac{1}{x} \right)
\]

Now take \( x \to \infty \) to get \(-2\)

\(\Box\)
7. If $\sum_{k=1}^{n} a_k = n^2 + n$, find $\sum_{k=11}^{20} a_k$.

(A) 42  
(B) 110  
(C) 132  
(D) 310  
(E) 420

\[
\sum_{k=11}^{20} a_k = \sum_{k=1}^{20} a_k + \sum_{k=1}^{10} a_k = (20^2 + 20) - (10^2 + 10) = 310\]

8. Let $f(x) = x^3 - 2$. Use Newton’s method to find a solution of $f(x) = 0$ beginning with $x_0 = 2$. Give a decimal approximation of $x_2$, correctly rounded to three decimal places.

(A) -2.444  
(B) 1.208  
(C) 1.260  
(D) 1.296  
(E) 1.500

\[
f'(x) = 3x^2
\]

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \left[\frac{x_n^3 - 2}{3x_n^2}\right]
\]
9. Let \( f(x) = x^3 \) on the interval \([-1, 1) = \{x : -1 \leq x < 1\} \). Find the absolute maximum and minimum values of \( f \) on the interval \([-1, 1)\).

(A) The absolute minimum value is 1 and the absolute maximum value is -1.
(B) The absolute minimum value is -1 and the absolute maximum value is 1.
(C) The absolute minimum value is -1 and the absolute maximum value is 0.
(D) The absolute minimum value is -1 and there is no absolute maximum value.
(E) There is no absolute minimum value and the absolute maximum value is 0.

10. Let \( f(x) = x^2 \). Divide the interval \([0,2] \) into three subintervals of equal length and compute \( R_3 \), the 3rd right-endpoint approximation to the area of the region \( R = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq x^2 \} \).

(A) 40/27
(B) 56/27
(C) 112/27
(D) 22/9
(E) 8/3

\[
R_3 = \frac{2}{3} \left[ f\left(\frac{2}{3}\right) + f\left(\frac{4}{3}\right) + f(2) \right]
\]

\[= \frac{2}{3} \left[ \frac{4}{9} + \frac{16}{9} + \frac{36}{9} \right] = \frac{2}{3} \cdot \frac{56}{9} = \frac{172}{27} \]
Free Response Questions: Show your work!

11. (a) State the mean value theorem.
   (b) For each function and interval determine if the mean value theorem applies. If
   the theorem does apply, state this. If the theorem does not apply, explain which
   hypothesis fails.
   i. $f(x) = x \sin(x)$ on the interval $[2, 42]$.
   ii. $g(x) = |x|$ on the interval $[-1, 1]$.

   **(a)** Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and differentiable
   on $[a, b]$. Then there exists at least one point
   $c$ with $a < c < b$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$.

   **(b) (i)** $f(x) = x \sin(x)$ on $[2, 42]$. $f$ is continuous everywhere
   and differentiable everywhere so the hypotheses on $f$ are satisfied. Also the interval
   $[2, 42]$ is closed and finite. So there is a $2 < c < 42$ where
   
   $f'(c) = \frac{f(42) - f(2)}{40} = \frac{42 \sin(42) - 2 \sin(2)}{40}$

   **(ii)** $g(x) = |x|$ on $[-1, 1]$

   $g$ is continuous on $[-1, 1]$ but not
   differentiable at $x = 0$. So the
   hypothesis of the MVT on $g$ is
   not satisfied & the MVT does not
   apply.
Free Response Questions: Show your work!

12. Let \( f(x) = e^{2x} \).

(a) Find \( L(x) \), the linearization of \( f(x) \) at 0. Put your answer in the form \( L(x) = mx + b \).

\[
L(x) : \quad \frac{f(x) - f(0)}{x - 0} = f'(0) = 2
\]

\( L(x) = 2x + 1 \)

(b) Find

\[
\lim_{x \to 0} \frac{f(x) - L(x)}{x^2}
\]

\[
L(x) = e^{2x}, \quad f(0) = 1
\]

\[
f'(x) = 2e^{2x}, \quad f'(0) = 2
\]

At \( x = 0 \), \( f(x) - L(x) = 1 - 1 = 0 \) so \( f(x) - L(x) \) is type indeterminate form at \( x = 0 \).

Apply 'Hôpital's Rule

\[
\lim_{x \to 0} \frac{f(x) - L(x)}{x^2} = \lim_{x \to 0} \frac{f'(x) - L'(x)}{2x}
\]

\[
= \lim_{x \to 0} \frac{2e^{2x} - 2}{2x} = \lim_{x \to 0} \frac{e^{2x} - 1}{x}
\]

Look at \( \frac{e^{2x}}{x} \) at \( x = 0 \). It is still \( \frac{0}{0} \) type.

Apply 'Hôpital's Rule again:

\[
\lim_{x \to 0} \frac{f(x) - L(x)}{x^2} = \lim_{x \to 0} \frac{2e^{2x}}{1} = \lim_{x \to 0} 2e^{2x} = 2.
\]
13. We have a rectangular piece of cardboard of area 200 cm². A box with no top is to be constructed by removing squares of side length 2 cm from each corner and folding up the remaining flaps.

(a) If the rectangle is $a$ cm $\times$ $b$ cm, find a function $V(a)$ which gives the volume of the box as a function of $a$. For which values of $a$ is it possible to construct a box?

(b) Find the dimensions $a$ and $b$ which give the box of largest volume and explain how you know you have found the largest volume.

\[ V(a) = 2 \left( b - 4 \right) \left( a - 4 \right) \]

**Constraint:**

- Clearly both $a$, $b > 4$.

**Substitute:**

\[ V(a) = 2 \left( \frac{200}{a} - 4 \right) \left( a - 4 \right) \]

\[ a \geq 4 \text{ and } b > 4 \implies a \leq 50 \text{ from } b = \frac{200}{a} \]

\[ a \in [4, 50] \]

**b.)** \[ V(a) = 2 \left( 200 - 4a - \frac{800}{a} + 16 \right) = 432 - 8a - \frac{1600}{a} \]

\[ V'(a) = -8 + \frac{1600}{a^2} = 0 \text{ so } a^2 = 200 \implies a = \sqrt[2]{200} = 10\sqrt{2} \geq 4. \]

2 ways to verify absolute max occurs at $a = 10\sqrt{2}$.

1. Check $V(a)$ at endpts. $V(4) = 0 = V(50)$ and $V(10\sqrt{2}) = 2(10\sqrt{2} - 4) > 0$

So this is the global max. $V(a) = b = 10\sqrt{2}$ cm

2. Note $V(a)$ is defined for $a > 0$

\[ V''(a) = -\frac{3200}{a^3} < 0 \implies \text{concave down} \]

So a global max at $10\sqrt{2}$. 
Free Response Questions: Show your work!

14. Suppose that a particle moves so that at time $t$ seconds, its acceleration is $a(t) = 6t - 2$ cm/second\(^2\). The position at time $t = 0$ is 7 cm to the right of the origin and the velocity at time $t = 1$ is 2 cm/second.

(a) Find a function which gives the position at all times $t$.

(b) Find the velocity at $t = 2$.

\[ V(t) = \int a(t) \, dt \quad \text{since} \quad V'(t) = a(t) \]

\[ V(t) = \int (6t-2) \, dt = 6 \int t \, dt - 2 \int dt = 3t^2 - 2t + C \quad (\text{cm/sec}) \]

Check: \[ V'(t) = 6t - 2 \quad \checkmark \]

Condition: \[ V(1) = 2 \text{ cm/sec} = 3 - 2 + C = 1 + C \Rightarrow C = 1 \]

\[ V(t) = 3t^2 - 2t + 1 \quad \text{cm/sec} \]

\[ S(t) = \int V(t) \, dt \quad \text{since} \quad S'(t) = V(t) \]

\[ S(t) = \int (3t^2 - 2t + 1) \, dt = t^3 - t^2 + t + D \]

Condition: \[ S(0) = 7 \text{ cm} = D \]

\[ S(t) = t^3 - t^2 + t + 7 \text{ cm} \quad \text{Check:} \quad S'(t) = 3t^2 - 2t + 1 \]

\[ S''(t) = 6t - 2 \quad \checkmark \]

(6) \[ V(t=2) = 3 \cdot 4 - 2 \cdot 2 + 1 = 9 \text{ cm/sec} \]
Free Response Questions: Show your work!

15. You may find one or more of the following formulae useful for this problem.

\[ \sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}. \]

Consider the sum

\[ S_N = \sum_{k=1}^{N} \left( 2 + \frac{3k}{N} \right). \]

(a) Find a closed form expression for \( S_N \).

(b) Find the limit \( \lim_{N \to \infty} \frac{1}{N} S_N \).

\[ \begin{align*}
(\text{a}) \quad S_N &= 2 \left( \sum_{k=1}^{N} 1 \right) + \frac{3}{N} \left( \sum_{k=1}^{N} k \right) \\
&= 2N + \frac{3}{N} \left( \frac{N(N+1)}{2} \right) \\
&= 2N + \frac{3}{2} (N+1) \\
&= 2N + \frac{3N}{2} + \frac{3}{2} N \\
&= \frac{3}{2} + \frac{7}{2} N.
\end{align*} \]

\[ S_N = \frac{3}{2} + \frac{7}{2} N \]

\[ \begin{align*}
(\text{b}) \quad \lim_{N \to \infty} \frac{1}{N} S_N &= \lim_{N \to \infty} \left( \frac{3}{2N} + \frac{7}{2} \right) \\
&= \frac{7}{2}.
\end{align*} \]

\[ \lim_{N \to \infty} \frac{1}{N} S_N = \frac{7}{2} \]