MA123, Chapter 8: Idea of the Integral (pp. 155-187, Gootman)

**Chapter Goals:**
- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Estimate the value of a definite integral.
- Understand the summation, or Σ, notation.
- Understand the formal definition of the definite integral.

**Assignments:** Assignment 18 Assignment 19

**The basic idea:** The first two problems are easy to solve as certain “problem ingredients” are constant.

**Example 1 (Easy area problem):** Find the area of the region in the xy-plane bounded above by the graph of the function \( f(x) = 2 \), below by the x-axis, on the left by the line \( x = 1 \), and on the right by the line \( x = 5 \).

\[
\text{Area} = 2 \cdot 4 = 8
\]

**Example 2 (Easy distance traveled problem):** Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

\[
55 \text{ mph} \cdot 2 \text{ hours} = 110 \text{ miles}
\]

**General philosophy:** By means of the integral, problems similar to the previous ones can be solved when the ingredients of the problem are variable. In this Chapter, we learn how to estimate a solution to these more complex problems. The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximately constant over a small interval. By the end of Chapter 9 we will be able to solve these problems exactly, and by the end of Chapter 10 we will be able to solve them both exactly and easily.

**Example 3:** Estimate the area under the graph of \( y = x^2 + \frac{1}{2}x \) for \( x \) between 0 and 2 in two different ways:

(a) Subdivide the interval \([0, 2]\) into four equal subintervals and use the left endpoint of each subinterval as “sample point”.

\[
\begin{align*}
\text{Area} &= \frac{1}{4} \left[ f(0) + f \left( \frac{1}{4} \right) + f \left( \frac{1}{2} \right) + f \left( \frac{3}{4} \right) \right] \\
&= \frac{1}{4} \left[ 0 + \frac{1}{4} + \frac{3}{8} + \frac{9}{8} \right] \\
&= \frac{25}{32} \\
&\approx 0.78125
\end{align*}
\]

(b) Subdivide the interval \([0, 2]\) into four equal subintervals and use the right endpoint of each subinterval as “sample point”.

\[
\begin{align*}
\text{Area} &= \frac{1}{4} \left[ f \left( \frac{1}{4} \right) + f \left( \frac{1}{2} \right) + f \left( \frac{3}{4} \right) + f(2) \right] \\
&= \frac{1}{4} \left[ 0 + \frac{5}{8} + \frac{9}{8} + \frac{8}{2} \right] \\
&= \frac{25}{32} \\
&= 0.78125
\end{align*}
\]

Find the difference between the two estimates (right endpoint estimate minus left endpoint estimate).

\[
5 - 2.5 = 2.5
\]
Example 5: Estimate the area of the ellipse given by the equation 
\[ 4x^2 + y^2 = 49 \]
as follows: The area of the ellipse is 4 times the area of the part of the ellipse in the first quadrant (x and y positive). Estimate the area of the ellipse in the first quadrant by solving for y in terms of x. Estimate the area under the graph of y by dividing the interval \([0, 3.5]\) into four equal subintervals and using the left endpoint of each subinterval.

\[
Area \text{ of approx. rectangle in first quadrant} = 0.875 \times (0) + 0.875 \times (0.875) + 0.875 \times (1.75) + 0.875 \times (2.625) \\
= 0.875 \left[ \frac{49 - 4 \cdot 0^2}{4} + \frac{49 - 4 \cdot (0.875)^2}{4} + \frac{49 - 4 \cdot (1.75)^2}{4} + \frac{49 - 4 \cdot (2.625)^2}{4} \right] \\
\approx 21.4112 \]

so Area whole Ellipse \( \approx 85.6448 \)

Trapezoids versus rectangles:

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. This will occur in some of the latter examples. We recall that the area of a trapezoid is

\[ \text{Area of a trapezoid} = \frac{(h_1 + h_2) \cdot b}{2} \]

Example 6: A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>0</td>
<td>10</td>
<td>15</td>
<td>18</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

We will see later that the total distance traveled by the train is equal to the area underneath the graph of the velocity function and lying above the t-axis. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of t on the subintervals. (The velocity in the table is given in miles per hour.)

Use trapezoids:

\[
(0.1) \left( \frac{0 + 10}{2} \right) + (0.1) \left( \frac{10 + 15}{2} \right) + (0.1) \left( \frac{15 + 18}{2} \right) + (0.1) \left( \frac{18 + 20}{2} \right) \\
+ (0.1) \left( \frac{20 + 25}{2} \right) = 7.57 \text{ miles} \]
Example 9: Evaluate the sum \( \sum_{k=2}^{6} (6k^2 + 3) \).

\[
= (6 \cdot 2^3 + 3) + (6 \cdot 3^3 + 3) + (6 \cdot 4^3 + 3) + (6 \cdot 5^3 + 3) + (6 \cdot 6^3 + 3)
\]

\[
= 6 \sum_{k=2}^{5} k^3 + 3(3^3 + 4^3 + 5^3 + 6^3)
\]

\[
= 6 \cdot 440 + 5 \cdot 3 = 2655
\]

Example 10: Evaluate the sum \( \sum_{k=1}^{5} (3k^2 + k) \).

\[
= (3 \cdot 1^2 + 1) + (3 \cdot 2^2 + 2) + (3 \cdot 3^2 + 3) + (3 \cdot 4^2 + 4) + (3 \cdot 5^2 + 5)
\]

\[
= 3(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + (1 + 2 + 3 + 4 + 5)
\]

\[
= 3 \cdot 55 + 15 = 180
\]

Example 11: Evaluate the sum \( \sum_{k=1}^{112} 75 \).

\[
= 75 + 75 + \ldots + 75 = 75 \times 112
\]

112 times

\[= 8400\]

Example 12: Evaluate the sum \( \sum_{k=15}^{273} 23 \).

\[
= 23 + 23 + \ldots + 23
\]

\[= (273 - 15) + 1 = 259 \]

\[
\sum_{k=15}^{273} 23 = 23 \cdot 259 = 5957
\]
**Right versus left endpoint estimates:**

Observe that \( x_k \), the right endpoint of the \( k \)-th subinterval, is also the left endpoint of the \((k+1)\)-th subinterval. Thus the Riemann sum estimate for the definite integral of a function \( f \) defined over an interval \([a, b]\) can be written in either of the following two forms

\[
\sum_{k=0}^{n-1} f(x_k) \cdot \Delta x_{k+1} \quad \text{or} \quad \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k
\]

depending on whether we use left or right endpoints, respectively.

If we are dealing with a regular partition, the above sums become

\[
\sum_{k=0}^{n-1} f(a + k \cdot \Delta x) \cdot \Delta x \quad \text{or} \quad \sum_{k=1}^{n} f(a + k \cdot \Delta x) \cdot \Delta x
\]

respectively, with \( \Delta x = (b - a) / n \) and \( x_k = a + k \cdot \Delta x \) for \( k = 0, 1, 2, \ldots, n \).

**Example 13:** Suppose you estimate the integral \( \int_{1}^{7} 8x \, dx \) by evaluating the sum

\[
\sum_{k=1}^{n} 8(1 + k \cdot \Delta x) \cdot \Delta x.
\]

If you use \( \Delta x = 0.2 \), what value should you use for \( n \), the upper limit of the summation?

\[
\Delta x = \frac{7 - 1}{n} \implies 0.2 = \frac{6}{n} \implies n = \frac{6}{0.2} = 30
\]

**Example 14:** Suppose you estimate the integral \( \int_{2}^{10} x^2 \, dx \) by evaluating the sum

\[
\sum_{k=1}^{n} (2 + k \cdot \Delta x)^2 \cdot \Delta x.
\]

If you use \( n = 10 \) intervals, what value should you use for \( \Delta x \), the length of each interval?

\[
\Delta x = \frac{10 - 2}{10} = \frac{8}{10} = 0.8
\]

**Example 15:** Suppose you estimate the integral \( \int_{-6}^{0} x^2 \, dx \) by the sum

\[
\sum_{k=1}^{n} [A + B(k \Delta x) + C(k \Delta x)^2] \cdot \Delta x,
\]

where \( n = 30 \) and \( \Delta x = 0.2 \). The terms in the sum equal areas of rectangles obtained by using right endpoints of the subintervals of length \( \Delta x \) as sample points. What is the value of \( B \)?

\[
\begin{align*}
X_k &= a + k \Delta x = -6 + k \Delta x, \text{ so sum will be } \left( \frac{k \Delta x}{12} \right) \\
\int_{-6}^{0} x^2 \, dx &= \frac{n}{2} \left( \frac{-6 + (k \Delta x)^2}{2} \right) \Delta x \\
&= \frac{n}{2} \left( \frac{-6 + (k \Delta x)^2}{2} \right) \Delta x \\
&= \frac{n}{2} \left( \frac{-6 + (k \Delta x)^2}{2} \right) \Delta x
\end{align*}
\]

where \( n = 30 \) and \( \Delta x = 0.2 \). The terms in the sum equal areas of rectangles obtained by using right endpoints of the subintervals of length \( \Delta x \) as sample points. What is the value of \( B \)?
Example 20: Suppose you estimate the area under the graph of \( f(x) = \frac{1}{x} \) from \( x = 12 \) to \( x = 112 \) by adding the areas of rectangles as follows: partition the interval into 50 equal subintervals and use the left endpoint of each interval to determine the height of the rectangle. What is the area of the 24th rectangle?

\[
\Delta x = \frac{12 - 12}{50} = 2 \\
A_k = f(a_k) \Delta x = 12 + 2k \\
\text{Area of k^{th} rectangle} = f(x_{k-1}) \Delta x \\
\text{since left endpoint} \\
\]

\[
f(x_{23}) \cdot \Delta x = \text{Area} \quad \text{of 24^{th} rectangle} = \frac{1}{29} \cdot 2 = \frac{1}{29}
\]

Example 21: Suppose you are given the following data points for a function \( f(x) \):

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

If \( f \) is a linear function on each interval between the given points, find \( \int_1^4 f(x) \, dx \).

Use trapezoids for area.

\[
\text{Area} = 1 \cdot \left( \frac{2 + 5}{2} \right) + 1 \cdot \left( \frac{5 + 8}{2} \right) + 1 \cdot \left( \frac{8 + 12}{2} \right) \\
= 20
\]

Example 22: Suppose \( f(x) \) is the greatest integer function, i.e., \( f(x) \) equals the greatest integer less than or equal to \( x \). So for example \( f(2.3) = 2 \), \( f(4) = 4 \), and \( f(6.9) = 6 \).

Find \( \int_6^{10} f(x) \, dx \).

The exact area \( \int_6^{10} f(x) \, dx \) is the exact area of 4 rectangles, each of width 1.

\[
\text{Area} = 1 \cdot f(6) + 1 \cdot f(7) + 1 \cdot f(8) + 1 \cdot f(9) \\
= 6 + 7 + 8 + 9 = 30
\]

(Hint: Draw a picture. See also example 18 in Chapter 1 and example 19 in Chapter 3.)