MA 137 – Calculus 1 with Life Science Applications

Discrete-Time Models

Sequences and Difference Equations: Limits
(Section 2.2)

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September 16, 2016
We now discuss how to find the limit when $a_n$ is defined by a recursive sequence of the first order

$$a_{n+1} = f(a_n)$$

Finding an explicit expression for $a_n$ is often not a feasible strategy, because solving recursions can be very difficult or even impossible.

How, then, can we say anything about the limiting behavior of a recursively defined sequence?

The following procedure will allow us to identify candidates for limits.
A fixed point (or equilibrium) of a recursive sequence is a number $a$ that is left unchanged by the (updating function) $f$, that is,

$$a = f(a)$$

Remark:
A fixed point is only a candidate for a limit; a sequence does not have to converge to a given fixed point (unless $a_0$ is already equal to the fixed point).
Example 1:

Let \( a_{n+1} = 1 + \frac{1}{a_n} \). Find the fixed points of this recursion, and investigate the limiting behavior of \( a_n \) when \( a_1 = 1 \).
Example 2:

Let \( a_{n+1} = \sqrt{3}a_n \). Find the fixed points of this recursion, and investigate the limiting behavior of \( a_n \) when \( a_0 = 1 \).
Example 3:

Let $a_{n+1} = \frac{3}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of $a_n$ when $a_0$ is not equal to a fixed point.
The previous examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence \( \{a_n\} \) may or may not converge to a given fixed point.

If we know, however, that a sequence \( \{a_n\} \) does converge, then the limit of the sequence must be one of the fixed points.

For this reason we say that a fixed point (or equilibrium) is **stable** if sequences that begin close to the fixed point approach that fixed point. It is called **unstable** if sequences that start close to the equilibrium move away from it.

We will return to the relationship between fixed points and limits in Section 5.6, where we will learn methods that allow us to determine whether a sequence converges to a particular fixed point.
A Graphical Way to Find Fixed Points

There is a graphical method for finding fixed points, which we mention briefly below.

Given a recursion of the form \( a_{n+1} = f(a_n) \), then we know that a fixed point \( \hat{a} \) satisfies \( \hat{a} = f(\hat{a}) \).

This suggests that if we graph \( y = f(x) \) and \( y = x \) in the same coordinate system, then fixed points are located where the two graphs intersect, as shown in the picture below.


Example 4:

(a) Consider the sequence recursively defined by the relation

\[ a_{n+1} = 2a_n(1 - a_n) \quad a_0 = 0 \]

and assume that \( \lim_{n \to \infty} a_n \) exists.

Find all fixed points of \( \{a_n\} \), and use a table or other reasoning to guess which fixed point is the limiting value for the given initial condition.

(b) Same as in (a) but with \( a_0 = 0.1 \).