Even and Odd Functions

Let $f$ be a function.

$f$ is **even** if $f(-x) = f(x)$ for all $x$ in the domain of $f$.

$f$ is **odd** if $f(-x) = -f(x)$ for all $x$ in the domain of $f$.

**Example:**

$y = \cos x$ is an even function; $y = \sin x$ is an odd function.

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**Example 1:**

Determine whether the following functions are even or odd:

$f(x) = x^3 + 2x^5$

$g(x) = x^2 - 3x^4$

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**Curious/Awesome Fact!**

Any function can be uniquely written as an even plus an odd function.

Example:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$

$$y = e^x$$

$$y = \cosh x$$

$$y = \sinh x$$
Example 2: (Online Homework HW02, #11)

(Since we talked about trigonometric functions...)

The lungs do not completely empty or completely fill in normal breathing. The volume of the lungs normally varies between 2140 ml and 2700 ml with a breathing rate of 22 breaths/min. This exchange of air is called the tidal volume.

One approximation for the volume of air in the lungs uses the cosine function written in the following manner:

\[ V(t) = A + B \cos(\omega t), \]

where \( A, B, \) and \( \omega \) are constants and \( t \) is in minutes. Use the data above to create a model, finding the constants \( A = \ldots \), \( B = \ldots \), and \( \omega = \ldots \), that simulates the normal breathing of an individual for one minute.

Combining functions

Let \( f \) and \( g \) be functions with domains \( A \) and \( B \). We define new functions \( f + g \), \( f - g \), \( fg \), and \( f/g \) as follows:

\[
(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B
\]
\[
(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B
\]
\[
(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B
\]
\[
\left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}
\]

Composition of Functions

Given any two functions \( f \) and \( g \), we start with a number \( x \) in the domain of \( g \) and find its image \( g(x) \). If this number \( g(x) \) is in the domain of \( f \), we can then calculate the value of \( f(g(x)) \).

The result is a new function \( h(x) = f(g(x)) \) obtained by substituting \( g \) into \( f \). It is called the composition (or composite) of \( f \) and \( g \) and is denoted by \( f \circ g \) (read: ‘\( f \) composed with \( g \)’ or ‘\( f \) after \( g \)’)

\[
(f \circ g)(x) \overset{\text{def}}{=} f(g(x)).
\]

**WARNING:** \( f \circ g \neq g \circ f \).
Example 3:

Let \( f(x) = \frac{x}{x+1} \) and \( g(x) = 2x - 1 \).

Find the functions \( f \circ g \), \( g \circ f \), and \( f \circ f \) and their domains.

Example 4:

Express the function \( F(x) = \frac{x^2}{x^2 + 4} \) in the form \( F(x) = f(g(x)) \).

Definition of a One-One Function

A function \( f \) with domain \( A \) is called a **one-to-one function** if no two elements of \( A \) have the same image, that is,

\[ f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2. \]

An equivalent way of writing the above condition is:

\[ f(x_1) = f(x_2) \quad \text{implies} \quad x_1 = x_2. \]

Horizontal Line Test

For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

**Horizontal Line Test**

A function is one-to-one if no horizontal line intersects its graph more than once.

A function is not one-to-one if a horizontal line intersects its graph more than once.
The Inverse of a Function

One-to-one functions are precisely those for which one can define a (unique) inverse function according to the following definition.

**Definition of the Inverse of a Function**

Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Its inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by

\[
f^{-1}(y) = x \iff f(x) = y,
\]
for any \( y \in B \).

If \( f \) takes \( x \) to \( y \), then \( f^{-1} \) takes \( y \) back to \( x \). I.e., \( f^{-1} \) undoes what \( f \) does.

**NOTE:**

\( f^{-1} \) does NOT mean \( \frac{1}{f} \).

**Properties of Inverse Functions**

Let \( f(x) \) be a one-to-one function with domain \( A \) and range \( B \). The inverse function \( f^{-1}(y) \) satisfies the following “cancellation” properties:

\[
f^{-1}(f(x)) = x \quad \text{for every } x \in A
\]

\[
f(f^{-1}(y)) = y \quad \text{for every } y \in B
\]

Conversely, any function \( f^{-1}(y) \) satisfying the above conditions is the inverse of \( f(x) \).

**Remark:**

Typically we write functions in terms of \( x \).
To do this, we need to interchange \( x \) and \( y \) in \( x = f^{-1}(y) \).

**Example 5:**

Show that the functions \( f(x) = x^5 \) and \( g(x) = x^{1/5} \) are inverses of each other.

1. Write \( y = f(x) \).
2. Solve this equation for \( x \) in terms of \( y \) (if possible).
3. Interchange \( x \) and \( y \). The resulting equation is \( y = f^{-1}(x) \).
Example 6: (Online Homework HW02, # 12)
Find the inverse of \( y = \frac{2 - 3x}{8 - 7x} \).

Example 7: (Exam 1, Spring 15, # 4)
One of the main quantities that epidemiologists try to measure for infectious diseases is the so-called basic reproduction number, \( R_0 \). Biologically, this is the expected number of new infections that an infected individual will produce when introduced into a completely susceptible population.

We can try to modify this by introducing vaccination to control the probability of an outbreak of the disease. We want to know the fraction of the population that we have to vaccinate to achieve a target outbreak probability. If \( v \) is the vaccination fraction, then the outbreak probability as a function of \( v \) is

\[
P = 1 - \frac{1}{R_0(1 - v)}.\]

Find the inverse of this function to obtain \( v \), the vaccination coverage needed, as a function of \( P \), the given target outbreak probability.

Example 8:
Find the inverse of the function \( f(x) = 1 + \sqrt{1+x} \).
Find the domain and range of \( f \) and \( f^{-1} \).
Graph \( f \) and \( f^{-1} \) on the same cartesian plane.