Exponential Functions

The exponential function

\[ f(x) = a^x \quad (a > 0, \ a \neq 1) \]

has domain \( \mathbb{R} \) and range \((0, \infty)\). The graph of \( f(x) \) has one of these shapes:

\[
\begin{align*}
\text{For } a > 1 & : & \quad & \text{growth} \\
\text{For } 0 < a < 1 & : & \quad & \text{decay}
\end{align*}
\]

Example 1:

Use the graph of \( f(x) = 3^x \) to sketch the graph of each function:

\[ g(x) = -3^x \]

\[ h(x) = 1 - 3^{-x} \]
The Number ‘e’ (Euler’s constant)

The most important base is the number denoted by the letter e.

The number e is defined as the value that \((1+\frac{1}{n})^n\) approaches as \(n\) becomes very large.

Correct to five decimal places (note that e is an irrational number), \(e \approx 2.71828\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>((1+\frac{1}{n})^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.00000</td>
</tr>
<tr>
<td>5</td>
<td>2.48832</td>
</tr>
<tr>
<td>10</td>
<td>2.59374</td>
</tr>
<tr>
<td>100</td>
<td>2.70481</td>
</tr>
<tr>
<td>1,000</td>
<td>2.71692</td>
</tr>
<tr>
<td>10,000</td>
<td>2.71815</td>
</tr>
<tr>
<td>100,000</td>
<td>2.71827</td>
</tr>
<tr>
<td>1,000,000</td>
<td>2.71828</td>
</tr>
</tbody>
</table>

Logarithmic Functions

Every exponential function \(f(x) = a^x\), with \(0 < a \neq 1\), is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the logarithmic function with base \(a\) and denoted by \(\log_a x\).

Definition

Let \(a\) be a positive number with \(a \neq 1\). The logarithmic function with base \(a\), denoted by \(\log_a x\), is defined by

\[ y = \log_a x \iff a^y = x. \]

That is, \(\log_a x\) is the exponent to which \(a\) must be raised to give \(x\).

Properties of Logarithms

1. \(\log_a 1 = 0\)
2. \(\log_a a = 1\)
3. \(\log_a a^x = x\)
4. \(a^{\log_a x} = x\)
Graphs of Logarithmic Functions

The graph of \( f^{-1}(x) = \log_a x \) is obtained by reflecting the graph of \( f(x) = a^x \) in the line \( y = x \). Thus, the function \( y = \log_a x \) is defined for \( x > 0 \) and has range equal to \( \mathbb{R} \).

![Graph of logarithmic function](image)

The point \((1, 0)\) is on the graph of \( y = \log_a x \) (as \( \log_a 1 = 0 \)) and the y-axis is a vertical asymptote.

Natural Logarithms

Of all possible bases \( a \) for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number \( e \).

Definition

The logarithm with base \( e \) is called the **natural logarithm** and denoted:

\[
\ln x := \log_e x.
\]

We recall again that, by the definition of inverse functions, we have

\[
y = \ln x \quad \iff \quad e^y = x.
\]

Properties of Natural Logarithms

1. \( \ln 1 = 0 \)
2. \( \ln e = 1 \)
3. \( \ln e^x = x \)
4. \( e^{\ln x} = x \)

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let \( a \) be a positive number, with \( a \neq 1 \). Let \( A \), \( B \) and \( C \) be any real numbers with \( A > 0 \) and \( B > 0 \).

1. \( \log_a(AB) = \log_a A + \log_a B \);
2. \( \log_a \left( \frac{A}{B} \right) = \log_a A - \log_a B \);
3. \( \log_a(A^C) = C \log_a A \).
Proof of Law 1.: \( \log_a(AB) = \log_a A + \log_a B \)

Let us set \( \log_a A = u \) and \( \log_a B = v \).
When written in exponential form, they become
\( a^u = A \) and \( a^v = B \).
Thus:
\[
\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})
\]
why?
\[
= u + v = \log_a A + \log_a B.
\]
In a similar fashion, one can prove 2. and 3.

Example 3:
Use the Laws of Logarithms to combine the expression
\[ \log_a b + c \log_a d - r \log_a s - \log_a t \]
into a single logarithm.

Proof of Law 1.: \( \log_a(AB) = \log_a A + \log_a B \)

Let us set \( \log_a A = u \) and \( \log_a B = v \).
When written in exponential form, they become
\( a^u = A \) and \( a^v = B \).
Thus:
\[
\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})
\]
why?
\[
= u + v = \log_a A + \log_a B.
\]
In a similar fashion, one can prove 2. and 3.

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

\[
\log_b x = \frac{\log_a x}{\log_a b}
\]

Proof: Set \( y = \log_b x \). By definition, this means that \( b^y = x \). Apply now \( \log_a(\cdot) \) to \( b^y = x \). We obtain
\[
\log_a(b^y) = \log_a x \quad \rightarrow \quad y \log_a b = \log_a x.
\]
Thus
\[
\log_b x = y = \frac{\log_a x}{\log_a b}.
\]

Example: \( \log_5 2 = \frac{\log 2}{\log 5} = \frac{\ln 2}{\ln 5} \approx 0.43068 \).
Example 4: (Online Homework HW03, # 6)

Solve the given equation for x:
\[ 2^{5x-4} = 3^{10x-10} \]

Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,
\[ \log_2(25 - x) = 3. \]

To solve for x, we write the equation in exponential form, and then solve for the variable:
\[ 25 - x = 2^3 \quad \Rightarrow \quad 25 - x = 8 \quad \Rightarrow \quad x = 17. \]

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:
\[ 2^{\log_2(25-x)} = 2^3 \quad \Rightarrow \quad 25 - x = 2^3 \quad \Rightarrow \quad x = 17. \]

Example 5: (Online Homework HW03, # 5)

Solve the given equation for x:
\[ \log_{10} x + \log_{10}(x + 21) = 2 \]