MA137 – Calculus 1 with Life Science Applications
Semilog and Double Log Plots
(Section 1.3)

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Logarithmic Scales

- When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers.
- Quantities that are measured on logarithmic scales include:
  - acidity of a solution (the pH scale),
  - earthquake intensity (Richter scale),
  - loudness of sounds (decibel scale),
  - light intensity,
  - information capacity,
  - radiation.
- In such cases, the equidistant marks on a logarithmic scale represent consecutive powers of 10.

The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, defined a more convenient measure:

$$\text{pH} = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter ($M$).

Solutions are defined in terms of the pH as follows:
- those with pH = 7 (or $[H^+] = 10^{-7}M$) are neutral,
- those with pH < 7 (or $[H^+] > 10^{-7}M$) are acidic,
- those with pH > 7 (or $[H^+] < 10^{-7}M$) are basic.

Example 1 (Finding pH):

The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}M$

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}M$

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}M$
Example 2 (Ion Concentration):

Calculate the hydrogen ion concentration of each substance from its pH reading.

(a) Vinegar: pH = 3.0

$$\text{Vinegar: } pH = 3.0 \Rightarrow 3.0 = -\log [H^+] \Rightarrow \log [H^+] = -3$$

$$\Rightarrow [H^+] = 10^{-3}$$

(b) Milk: pH = 6.5

$$\text{Milk: } pH = 6.5 \Rightarrow 6.5 = -\log [H^+] \Rightarrow \log [H^+] = -6.5$$

$$\Rightarrow [H^+] = 10^{-6.5} = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10^{-0.5} = 3.2 \times 10^{-7}$$

Semilog Plots

- In biology it is common to use a semilog plot to see whether data points are appropriately modeled by an exponential function.
- This means that instead of plotting the points $(x, y)$, we plot the points $(x, \log y)$.
- In other words, we use a logarithmic scale on the vertical axis.
The graphs are taken from the article


David Ho was Time magazine’s 1996 Man of the Year.

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**How to Read a Semilog Plot**

You need remember is that the log axis runs in exponential cycles. Each cycle runs linearly in 10’s but the increase from one cycle to another is an increase by a factor of 10. So within a cycle you would have a series of: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (this could also be 0.1-1, etc.). The next cycle begins with 10 and progresses as 20, 30, 40, 50, 60, 70, 80, 90, 100. The cycle after that would be 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000.

Below is a picture of semilog graph paper.

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**Example 3:**

Suppose that $x$ and $y$ are related by the expression

$$y = 4 \cdot 10^{-x/2} = 4 \cdot (10^{-1/2})^x = 4 \cdot (0.316)^x.$$  

Use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship in the semilog (or log-linear) plot.

Let’s plot a few values of the function:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$10^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>0.0004</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
<td>0.04</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1.265</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>0.4</td>
<td>100</td>
</tr>
</tbody>
</table>

From $y = 4 \cdot 10^{-x/2}$ take log of both sides:

$$\log y = \log (4 \cdot 10^{-x/2})$$
$$= \log (4) + \log (10^{-x/2})$$
$$= \log (4) + (-x/2) \cdot x.$$  

Set $Y = \log y$, so the above equation becomes:

$$Y = (-\frac{1}{2}) \cdot x + \log (4)$$  

Note that the intercept is plotted at $(\log(4), 0)$. 

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Logarithmic Scales

The pH Scale

Semilog Plots

Logarithmic Scales

The pH Scale

Semilog Plots
Lines in Semilog Plots

- If we start with an exponential function of the form $y = a \cdot b^x$ and take logarithms of both sides, we get
  \[ \log y = \log(a \cdot b^x) = \log a + \log b^x \]
  \[ \log y = \log a + x \log b \]
  If we let $Y = \log y$, $M = \log b$, and $B = \log a$, then we obtain
  \[ Y = B + Mx, \]
  i.e., the equation of a line with slope $M$ and $Y$-intercept $B$.

- So if we obtain experimental data that we suspect might possibly be exponential, then we could graph a semilog scatter plot and see if it is approximately linear.

Example 4:

When $\log y$ is graphed as a function of $x$, a straight line results.
Graph the straight line given by the following two points

$$(x_1, y_1) = (0, 40) \quad (x_2, y_2) = (2, 600)$$

on a log-linear plot. Determine the functional relationship between $x$ and $y$. (Note: The original $x$-$y$ coordinates are given.)

Conversely, suppose we have a straight line in a semilog plot:

$$Y = Mx + B \quad \text{when} \quad Y = \log y$$

Then from $\log y = Mx + B$ we obtain

$$\log y = 10$$

\[ \iff \]

$$y = 10^M \cdot 10^B$$

\[ \iff \]

$$y = \left(\frac{10^B}{a}\right) \cdot \left(\frac{10^M}{b}\right)^x = a \cdot b^x$$

when $a = 10^8$ $b = 10^M$

First method: a line in a semilog plot corresponds to an exponential function of the form $y = a \cdot b^x$.

when $x = 0$ then $y = 40$

$x = 2$ then $y = 600$

\[ \begin{align*}
  600 & = a \cdot b^2 \\
  40 & = a \cdot b^0 \implies a = 40
\end{align*} \]

\[ \implies \begin{align*}
  a & = 40 \quad \text{and} \quad 600 = 40 \cdot b^2 \implies b^2 = \frac{600}{40} = 15 \\
  b & = \sqrt{15} \approx 3.873
\end{align*} \]

\[ \implies \begin{align*}
  y & = 40 \cdot (3.873)^x
\end{align*} \]

Second method: in the $(x, \log y)$ plot we need to compute the equation of the line through $(0, \log 40)$ and $(2, \log 600)$
When \( \log y \) is graphed as a function of \( x \), a straight line results. Graph the straight line given by the following two points

\[
( x_1, y_1 ) = (1, 4) \quad ( x_2, y_2 ) = (6, 1)
\]
on a log-linear plot. Determine the functional relationship between \( x \) and \( y \). (Note: The original \( x-y \) coordinates are given.)
Let’s compute the equation of the line in the semi-log plot through \((1, \log 4)\) and \((6, \log 1)\).

\[
m = \frac{\log 1 - \log 4}{6 - 1} = \frac{-\log 4}{5} = (-\frac{1}{5}) \log 4 = \log (4^{-\frac{1}{5}}) = \log \left(\frac{1}{\sqrt[5]{4}}\right)
\]

Hence

\[
y - \log 1 = \log \left(\frac{1}{\sqrt[5]{4}}\right) (x - 6)
\]

\[
\log y = (x - 6) \log \left(\frac{1}{\sqrt[5]{4}}\right) \quad \Rightarrow \quad y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x - 6}
\]

\[
\log y = \log \left[\left(\frac{1}{\sqrt[5]{4}}\right)^{x - 6}\right] \quad \Rightarrow \quad y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x - 6} = (5.278 (0.7578)^x)
\]

**Example 6:** (Problem # 52, Section 1.3, p. 53)

Consider the relationship \(y = 6 \times 2^{-0.9x}\) between the quantities \(x\) and \(y\). Use a logarithmic transformation to find a linear relationship of the form

\[Y = mx + b\]

between the given quantities.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = 6 \times 2^{-0.9x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>20.89</td>
</tr>
<tr>
<td>-1</td>
<td>11.196</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>3.215</td>
</tr>
<tr>
<td>2</td>
<td>1.423</td>
</tr>
</tbody>
</table>

\[
\log y = \log (6 \times 2^{-0.9x}) = \log 6 + \log (2^{-0.9x}) = (-0.9) \log 2 \times x + \log 6
\]

\[
\log y = -0.27092x + 0.7782
\]
approx. graph of $y = 6 \cdot 2^{-0.9x}$ in semi-log plot