MA137 – Calculus 1 with Life Science Applications

Discrete-Time Models
Sequences and Difference Equations
(Sections 2.1 and 2.2)

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What are sequences?

So far we have studied real valued functions whose domain consists of the real numbers, say:

\[ f : \mathbb{R} \rightarrow \mathbb{R}. \]

For example, consider the function

\[ f(t) = 3 \cdot 2^t. \]

The graph of \( f \) looks like:

More generally, we have considered functions of the form

\[ P(t) = P_0 (1 + r)^t, \]

where \( r \) is a positive real number (\( r \equiv \) growth rate).

Definition and Notation

Definition (Sequence/Notation)

We can write the function

\[ f : \mathbb{N} \rightarrow \mathbb{R}, \quad n \mapsto f(n) \]

as a list of numbers

\[ f_0, f_1, f_2, f_3, \ldots, \]

where \( f_n = f(n) \).

We refer to this list as a sequence.

We write \( \{f_n \mid n \in \mathbb{N}\} \) (or \( \{f_n\} \) for short) to denote the entire sequence.

We list the values of the sequence \( \{f_n\} \) in order of increasing \( n \)

\[ f_0, f_1, f_2, f_3, \ldots. \]

Remark: Instead of ‘f’ we often use the letters ‘a’ or ‘b’ or ‘c’ ... to denote sequences.

For example:

\[ a_n = \frac{n}{n+1}, \quad b_n = \frac{(-1)^n}{(n+1)^2}, \quad c_n = 3 \cdot 2^n \]
Example 1:

Find a general formula for the general term $a_n$ for each of the following sequences starting with $a_0$:

(a) $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

(b) $1, -1, 1, -1, 1, \ldots$

(c) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Repeat this problem starting this time with $a_1$.

(a) Consider $0, 1, 4, 9, 16, 25, 36, 49, \ldots$.

There are all squares of numbers.

We want them to be labeled as

\[ a_0 = 0, \ a_1 = 1, \ a_2 = 4, \ a_3 = 9, \ a_4 = 16, \ldots \]

Thus \[ a_n = n^2 \] is the $n$th term of the sequence.

(b) We want: $a_0 = 1, \ a_1 = -1, \ a_2 = 1, \ a_3 = -1, \ldots$

So we have \[ a_n = (-1)^n \] for all $n \in \mathbb{N}$.

(c) We want: $a_0 = 1, \ a_1 = -\frac{1}{2}, \ a_2 = \frac{1}{4}, \ a_3 = -\frac{1}{8}, \ldots$

\[ a_4 = \frac{1}{16}, \ \text{etc...} \]

Notice that all denominators are powers of 2; there is an alternating sign.

\[ a_n = \left(\frac{-1}{2}\right)^n \]

Example 2:

Consider the sequence given by

\[ a_n = 2 + \frac{(-1)^n}{n} \quad n > 1. \]

List the first six terms of the sequence and plot them on the Cartesian plane.
Recursions (or Recursive Sequences)

The exponential growth model we considered earlier
\[ P_n = 3 \cdot 2^n \]

is an example of a sequence. Explicitly, we have
\[ P_0 = 3, \quad P_1 = 6, \quad P_2 = 12, \quad P_3 = 24, \quad P_4 = 48, \quad \ldots \]

It is not difficult to observe that this sequence of numbers describes quantities that double after each unit of time.

More explicitly, we can write
\[ P_1 = 2P_0, \quad P_2 = 2P_1, \quad P_3 = 2P_2, \quad P_4 = 2P_3, \quad \ldots \]

We can summarize the above facts into a single expression. I.e.,
\[ P_{n+1} = 2P_n \]

this expression gives a rule that is applied repeatedly to go from one time step (the nth) to the next one (the (n+1)st).

Such an expression is called a recursion.

Example 3:

(a) List the first five terms of the recursively define sequence
\[ a_0 = 1 \quad a_{n+1} = (n+1)a_n. \]

Do you see something familiar?

(b) List the first five terms of the recursively define sequence
\[ a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{1}{a_n}. \]

Do you see something familiar?

Caution: While it is easy to compute terms in a recursive relation, there are 2 issues:
- In order to find \( a_{100} \), we have to compute the previous 99 terms.
- We may not get a feeling for what will eventually happen.
### Example 4: (Online Homework HW06, # 8)

(a) Find a recursive definition for the sequence \(9, 11, 13, 15, 17, \ldots\) Assume the first term in the sequence is indexed by \(n = 1\).

(b) Find a closed formula for the sequence \(9, 11, 13, 15, 17, \ldots\) Assume the first term in the sequence is indexed by \(n = 1\).

This sequence is given by the quotient of 2 consecutive Fibonacci's numbers.

When \(n \to \infty\) this ratio tends to \(1.618 = \frac{1 + \sqrt{5}}{2}\), \text{GOLDEN RATIO}.

### Recap

We gave two descriptions of sequences: explicit and recursive.

- An **explicit description** is of the form \(a_n = f(n)\), \(n = 0, 1, 2, \ldots\) where \(f(n)\) is a function of \(n\).

- A **recursive description** is of the form \(a_{n+1} = g(a_n)\), \(n = 0, 1, 2, \ldots\) where \(g(a_n)\) is a function of \(a_n\).

**Remark 1:**
In the above situation the value of \(a_{n+1}\) depends only on the value one time step back, namely, \(a_n\). In this case the recursion is called a **first-order recursion**.

**Remark 2:**
The sequence defined by \(a_0 = 1, a_1 = 1, a_{n+2} = a_n + a_{n+1}\) for \(n = 0, 1, 2, \ldots\) is an example of a **second-order recursion**.
Recursive Sequences in the Life Sciences

Recursive sequences (or **difference equations**) are often used in biology to model, for example, cell division and insect populations. In this biological context we usually replace \( n \) by \( t \), to denote time. If we think of \( t \) as the current time, then \( t + 1 \) is one unit of time into the future. We also use \( N_t \) to denote the population size. Thus a first-order difference equation modeling population size has the form

\[
N_{t+1} = f(N_t) \quad t = 0, 1, 2, 3, \ldots
\]

In this context we call \( f \) an **updating function** because \( f \) ‘updates’ the population from \( N_t \) to \( N_{t+1} \).

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**Example 5:** (Online Homework HW06, #11)

(a) A population of herbivores satisfies the growth equation

\[
y_{n+1} = 1.05y_n
\]

where \( n \) is in years. If the initial population is \( y_0 = 6,000 \), then determine the explicit expression of the population.

(b) A competing group of herbivores satisfies the growth equation

\[
z_{n+1} = 1.06z_n
\]

If the initial population is \( z_0 = 3,200 \), then determine how long it takes for this population to double.

(c) Find when the two populations are equal.

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Malthusian (or Exponential) Growth Model

One of our earlier examples can be rewritten as

\[
N_{t+1} = 2N_t \quad N_0 = 3 \quad \text{or} \quad N_t = 3 \cdot 2^t.
\]

This example is a special case of the so-called **Malthusian Growth Model**, named after Thomas Malthus (1766-1834):

\[
N_{t+1} = (1 + r)N_t
\]

which says that the next generation is proportional to the population of the current generation.

It is typical to set \( R = 1 + r \) so that the recursion becomes

\[
N_{t+1} = RN_t.
\]

This recursion has the following explicit form

\[
N_t = N_0 R^t.
\]

Hence the name of Exponential Growth Model.

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(a) \( y_n = 6,000 \cdot (1.05)^n \)

(b) \( z_n = 3,200 \cdot (1.06)^n \)

We want to know \( n \) such that

\[
3,200 \cdot (1.06)^n = 2 \cdot 3,200
\]

i.e. we want \((1.06)^n = 2\)

Take \( \log \) (or \( \ln \)) of both sides

\[
\log (1.06)^n = \log (2) \quad \Rightarrow \quad n = \frac{\log 2}{\log (1.06)} \approx 11.895
\]
Visualizing Recursions

We can visualize recursions by plotting $N_t$ on the horizontal axis and $N_{t+1}$ on the vertical axis. Since $N_t \geq 0$ for biological reasons, we restrict the graph to the first quadrant.

The exponential growth recursion

$$N_{t+1} = RN_t$$

is then a straight line through the origin with slope $R$.

[i.e., $N_{t+1} = f(N_t)$, where $f(x) = Rx$]

For any current population size $N_t$, the graph allows us to find the population size in the next time step, namely, $N_{t+1}$.

Unless we label the points according to the corresponding $t$-value, we would not be able to tell at what time a point $(N_t, N_{t+1})$ was realized. We say that time is implicit in this graph.

The hallmark of exponential growth is that the ratio of successive population sizes, $N_t/N_{t+1}$, is constant. More precisely, it follows from $N_{t+1} = RN_t$ that

$$\frac{N_t}{N_{t+1}} = \frac{1}{R}$$

If the population consists of annual plants, we can interpret the ratio $N_t/N_{t+1}$ as the parent-offspring ratio.

If this ratio is constant, parents produce the same number of offspring, regardless of the current population density. Such growth is called density independent.

When $R > 1$, the parent-offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. This model yields then an ever-increasing population size. It eventually becomes biologically unrealistic, since any population will sooner or later experience food or habitat limitations that will limit its growth.