MA 137 – Calculus 1 with Life Science Applications  
**Discrete-Time Models**  
Sequences and Difference Equations: **Limits**  
(Section 2.2)

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**Long-Term Behavior**

When studying populations over time, we are often interested in their long-term behavior.

Specifically, if $N_t$ is the population size at time $t$, $t = 0, 1, 2, \ldots$, we want to know how $N_t$ behaves as $t$ increases, or, more precisely, as $t$ tends to infinity.

Using our general setup and notation, we want to know the behavior of $a_n$ as $n$ tends to infinity and use the shorthand notation

$$\lim_{n \to \infty} a_n$$

which we read as ‘the limit of $a_n$ as $n$ tends to infinity.’

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**Definition and Notation**

**Definition (Informal)**

We say that the limit as $n$ tends to infinity of a sequence $a_n$ is a number $L$, written as $\lim_{n \to \infty} a_n = L$, if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large.

**Definition (Formal)**

The sequence $\{a_n\}$ has a limit $L$, written as $\lim_{n \to \infty} a_n = L$, if, for any given any number $d > 0$, there is an integer $N$ so that

$$|a_n - L| < d$$

whenever $n > N$.

If the limit exists, the sequence **converges** (or is **convergent**).  
Otherwise we say that the sequence **diverges** (or is **divergent**).
Example 1:

Let \( a_n = \frac{1}{n} \) for \( n = 1, 2, 3, \ldots \).

Show that \( \lim_{n \to \infty} \frac{1}{n} = 0 \).

Example 2:

Let \( a_n = (-1)^n \) for \( n = 0, 1, 2, \ldots \).

Show that \( \lim_{n \to \infty} (-1)^n \) does not exist.

What about the limit of the sequence \( b_n = \cos(\pi n) \)?
(a) \[ \lim_{n \to \infty} \frac{n(1-3n^2)}{n^2+1} = \lim_{n \to \infty} \frac{n}{n^2} \cdot \lim_{n \to \infty} (1-3n^2) = \lim_{n \to \infty} \frac{1}{n} \cdot \lim_{n \to \infty} 1-3n^2 \]

\[ = \frac{\infty}{\infty} \cdot (-\infty) = \infty \cdot (-\infty) = \text{which is not defined.} \]

However, notice that \[ \lim_{n \to \infty} \frac{1}{n} = 0 \]

\[ \Rightarrow \lim_{n \to \infty} \frac{1}{n^p} = 0 \quad \text{for any } p > 1 \]

Thus, we can rewrite our original limit as
\[
\lim_{n \to \infty} \frac{n(1 - 3n^2)}{n^3 + 1} = \lim_{n \to \infty} \frac{n - 3n^3}{n^3 + 1} = \\
= \lim_{n \to \infty} \frac{(n - 3n^3) \cdot \frac{1}{n^3}}{(n^3 + 1) \cdot \frac{1}{n^3}} = \lim_{n \to \infty} \left( \frac{\frac{1}{n^2} - 3}{1 + \frac{1}{n^3}} \right) \\
\text{Use now the properties of limits:} \\
= \lim_{n \to \infty} \left( \frac{\frac{1}{n^2} - 3}{1 + \frac{1}{n^3}} \right) = \left[ \lim_{n \to \infty} \left( \frac{1}{n^2} \right) \right] - \left[ \lim_{n \to \infty} \frac{3}{1 + \frac{1}{n^3}} \right] \\
= \lim_{n \to \infty} \left( \frac{1}{n^2} \right) + \lim_{n \to \infty} \frac{3}{1 + \frac{1}{n^3}} \\
= \frac{0 - 3}{1 + 0} = \frac{-3}{1} = -3
\]

(b) \[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = \frac{\lim_{n \to \infty} n}{\lim_{n \to \infty} (n^2 + 1)} = \frac{\infty}{\infty}
\]

However, we can rewrite this limit as:
\[
\lim_{n \to \infty} \frac{n}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} \\
= \lim_{n \to \infty} \frac{1}{1 + 0} = \frac{0}{1} = 0
\]

Can you see a general rule?

**Example 4:**

For \( R > 0 \), we know that exponential growth is given by

\[ N_t = N_0 R^n \quad n = 0, 1, 2, \ldots \]

The figure below indicates that
\[
\lim_{n \to \infty} N_t = \begin{cases} 
0 & \text{if } 0 < R < 1 \\
N_0 & \text{if } R = 1 \\
\infty & \text{if } R > 1
\end{cases}
\]

**Example 5:**

Find \( \lim_{n \to \infty} \frac{3 \cdot 4^n + 1}{4^n} \)
The values in the following table and the graph on the left suggest that for $n \geq 4$ we have

$$\frac{-1}{2^n} \leq \frac{(-1)^n}{n!} \leq \frac{1}{2^n} \quad n \geq 4.$$  

So by the Squeeze Theorem it follows that

$$\lim_{n \to \infty} \frac{(-1)^n}{n!} = 0.$$

**Example 6:**

Find $\lim_{n \to \infty} \frac{2n + (-1)^n}{n}$.
\[ b_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n} \]

Observe that \(-1 \leq (-1)^n \leq 1\) for every \(n\).

Thus,

\[
\begin{align*}
\frac{2 - 1}{n} \leq \frac{2 + (-1)^n}{n} & \leq \frac{2 + 1}{n} \\
\Rightarrow a_n & \leq \frac{2 + (-1)^n}{b_n} \leq c_n
\end{align*}
\]

and \(\lim_{n \to \infty} \left[ \frac{2 - 1}{n} \right] = 2 = \lim_{n \to \infty} \left[ \frac{2 + 1}{n} \right] \)

so that \(\lim_{n \to \infty} \frac{2n + (-1)^n}{n} = 2\)

Observe that

\[ 0 \leq \frac{5^n}{n!} \leq \frac{5 \times 5 \times 5 \ldots}{n!} \frac{5}{n} \frac{5}{n-1} \frac{5}{n-2} \ldots \frac{5}{2} \times \frac{5}{1} \]

we can reorganize these terms as

\[
\left[ \frac{5^n}{n!} \frac{5}{n} \frac{5}{n-1} \frac{5}{n-2} \ldots \frac{5}{n-5} \right] \left[ \frac{5}{5} \frac{5}{4} \frac{5}{3} \frac{5}{2} \frac{5}{1} \right] \leq \left( \frac{5}{6} \right)^{n-5} \cdot \frac{625}{24}
\]

In other words:

\[ 0 \leq \frac{5^n}{n!} \leq \left( \frac{5}{6} \right)^{n-5} \cdot \frac{625}{24} \]

But \(\lim_{n \to 0} 0 = 0 = \lim_{n \to \infty} \left( \frac{5}{6} \right)^{n-5} \cdot \frac{625}{24} \)

So \(\lim_{n \to \infty} \frac{5^n}{n!} = 0\) as \(\frac{5}{6} < 1\)