Limits of Recursive Sequences

We now discuss how to find the limit when $a_n$ is defined by a recursive sequence of the first order

$$a_{n+1} = f(a_n)$$

Finding an explicit expression for $a_n$ is often not a feasible strategy, because solving recursions can be very difficult or even impossible.

How, then, can we say anything about the limiting behavior of a recursively defined sequence?

The following procedure will allow us to identify candidates for limits.

Example 1:

Let $a_{n+1} = 1 + \frac{1}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of $a_n$ when $a_1 = 1$. 

Definition

A fixed point (or equilibrium) of a recursive sequence

$$a_{n+1} = f(a_n)$$

is a number $\hat{a}$ that is left unchanged by the (updating function) $f$, that is,

$$\hat{a} = f(\hat{a})$$

Remark:

A fixed point is only a candidate for a limit; a sequence does not have to converge to a given fixed point (unless $a_0$ is already equal to the fixed point).
Consider the recursive sequence \( a_{n+1} = 1 + \frac{1}{a_n} \)

(Notice that \( a_{n+1} = f(a_n) \) where \( f(x) = 1 + \frac{1}{x} \))

To find the fixed points we need to solve for \( a \) in:

\[ a = 1 + \frac{1}{a} \]

Multiply both sides by \( a \):

\[ a^2 = a + 1 \]

\[ a^2 - a - 1 = 0 \]

and use now the quadratic formula:

\[ a_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \]

Thus there are two fixed points:

\[ a_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618 \]

\[ a_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618 \]

\[ \text{GOLDEN RATIO} \]

Let's investigate \( \lim_{n \to \infty} a_n \).

We already worked out a few terms of this sequence in an earlier lecture:

\[
\begin{align*}
    a_1 &= 1 \\
    a_2 &= 1 + \frac{1}{a_1} = 1 + 1 = 2 \\
    a_3 &= 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = 3/2 = 1.5 \\
    a_4 &= 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3} = 1.6666... \\
    a_5 &= 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + \frac{3}{5} = \frac{8}{5} = 1.6 \\
    a_6 &= 1 + \frac{1}{a_5} = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \frac{13}{8} = 1.625 \\
\end{align*}
\]

We realize that the \( n \)-th term of the sequence \( a_n \) is the quotient of two consecutive Fibonacci's numbers \((1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)\)

From the first few terms of this sequence we have worked out \( a_1, a_2, \ldots, a_6 \) it seems obvious that \( \lim_{n \to \infty} a_n = \hat{a}_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618 \)

[Aside] it takes quite some work and some mathematical skill to prove that there exists an explicit form of the Fibonacci's numbers.

Namely:

\[
F_n = \frac{1}{\sqrt{5}} \left[ (\frac{1 + \sqrt{5}}{2})^n - (\frac{1 - \sqrt{5}}{2})^n \right]
\]

\[ \text{for } n = 1, 2, 3, \ldots \]

Let \( a_{n+1} = \sqrt{3}a_n \). Find the fixed points of this recursion, and investigate the limiting behavior of \( a_n \) when \( a_0 = 1 \).
Example 3:

Let \( a_{n+1} = \frac{3}{a_n} \). Find the fixed points of this recursion, and investigate the limiting behavior of \( a_n \) when \( a_0 \) is not equal to a fixed point.

\[
a_{n+1} = \sqrt{3}a_n
\]

(notice than \( a_{n+1} = f(a_n) \) when \( f(x) = \sqrt{3}x \))

To find the fixed points we have to solve

\[
a = \sqrt{3}a \implies a^2 = (\sqrt{3}a)^2 \quad [\text{i.e. we squared both sides}]
\]

\[
\iff a^2 = 3a \iff a^2 - 3a = 0
\]

\[
\iff a(a-3) = 0 \iff \begin{cases} \hat{a}_1 = 0 & \hat{a}_2 = 3 \end{cases}
\]

We want to investigate \( \lim_{n \to \infty} a_n \) with \( a_0 = 1 \)

Then

\[
a_0 = 1 ; \quad a_1 = \sqrt{3}a_0 = \sqrt{3} \approx 1.732 \quad ;
\]

\[
a_2 = \sqrt{3}a_1 = 2.279 ; \quad a_3 = \sqrt{3}a_2 = 2.615
\]

\[
a_4 = \sqrt{3}a_3 = 2.8 \quad ; \quad a_5 = \sqrt{3}a_4 = 2.898
\]

\[
a_6 = \sqrt{3}a_5 \approx 2.949 ; \quad \text{etc}....
\]

Hence all these calculations seem to suggest that

\[
\lim_{n \to \infty} a_n = 3
\]

that is, the limit is the fixed point \( \hat{a}_2 = 3 \).

\[
a_{n+1} = \frac{3}{a_n}
\]

[that is, \( a_{n+1} = f(a_n) \) with \( f(x) = \frac{3}{x} \)]

Fixed point: we need to solve the equation

\[
a = \frac{3}{a} \iff a^2 = 3 \iff a = \pm \sqrt{3}
\]

Thus there are two fixed points: \( \hat{a}_1 = \sqrt{3} ; \hat{a}_2 = -\sqrt{3} \)

(1) Suppose that \( a_0 = \sqrt{3} \Rightarrow a_1 = \frac{3}{a_0} = \frac{3}{\sqrt{3}} = \sqrt{3} \)

\[
a_2 = \frac{3}{\hat{a}_1} = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \text{hence} \quad a_n = \sqrt{3} \quad \text{for all} \; n.
\]

(2) Similarly, if we start with \( a_0 = -\sqrt{3} \) we get that

\[
a_1 = \frac{3}{a_0} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \quad ; \quad a_2 = \frac{3}{a_1} = -\frac{3}{\sqrt{3}} = -\sqrt{3}
\]

i.e. \( a_n = -\sqrt{3} \quad \text{for all} \; n \).
(3) However, let’s start for example with $a_0 = 2$. We have $a_1 = \frac{3}{a_0} = \frac{3}{2} = 1.5$
$a_2 = \frac{3}{a_1} = \frac{3}{1.5} = 2$; $a_3 = \frac{3}{a_2} = \frac{3}{2} = 1.5$; ....

Hence we conclude that even if we started close to the fixed point $\hat{a} = \sqrt{3}$, i.e., we picked $a_0 = 2$, we get
$a_0 = a_2 = a_4 = a_6 = a_8 = ... = 2$
$a_1 = a_3 = a_5 = a_7 = a_9 = ... = \frac{3}{2}$

Hence $\lim_{n \to \infty} a_n = \text{does not exist}$

Comments

The previous examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not converge to a given fixed point.

If we know, however, that a sequence $\{a_n\}$ does converge, then the limit of the sequence must be one of the fixed points.

For this reason we say that a fixed point (or equilibrium) is **stable** if sequences that begin close to the fixed point approach that fixed point. It is called **unstable** if sequences that start close to the equilibrium move away from it.

We will return to the relationship between fixed points and limits in Section 5.6, where we will learn methods that allow us to determine whether a sequence converges to a particular fixed point.

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A Graphical Way to Find Fixed Points

There is a graphical method for finding fixed points, which we mention briefly below.

Given a recursion of the form $a_{n+1} = f(a_n)$, then we know that a fixed point $\hat{a}$ satisfies $\hat{a} = f(\hat{a})$.

This suggests that if we graph $y = f(x)$ and $y = x$ in the same coordinate system, then fixed points are located where the two graphs intersect, as shown in the picture below.

![Graph of y = f(x) and y = x intersecting to find fixed points.](image)

Example 4:

(a) Consider the sequence recursively defined by the relation $a_{n+1} = 2a_n(1 - a_n)$, $a_0 = 0$
and assume that $\lim_{n \to \infty} a_n$ exists.
Find all fixed points of $\{a_n\}$, and use a table or other reasoning to guess which fixed point is the limiting value for the given initial condition.

(b) Same as in (a) but with $a_0 = 0.1$. 

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Notice that \( a_{n+1} = 2a_n(1-a_n) \) is of the form

\[ a_{n+1} = f(a_n) \text{ where } f(x) = 2x(1-x) \]

This is a parabola with a downward concavity.

To find the fixed points we need to solve

\[ a = 2a(1-a) \implies a = 0 \quad \text{or} \quad 1 = 2(1-a) \implies \frac{1}{2} = 1-a \implies a = 1 - \frac{1}{2} = \frac{1}{2} \]

Thus the fixed points are:

\[ a_1 = 0 \quad \text{or} \quad a_2 = \frac{1}{2} \]

Notice that the fixed points are geometrically given by the intersection points between

\[ y = f(x) = 2x(1-x) \text{ and } y = x \]

Thus: 
\[ \lim_{n \to \infty} a_n \]

(1) if \( a_0 = 0 \) \quad \( a_1 = 2a_0(1-a_0) = 0 \) \quad \( a_2 = 2a_1(1-a_1) = 0 \) \quad etc...

\[ \text{so } \lim_{n \to \infty} a_n = 0 \]

(2) Let's consider the case \( a_0 = 0.1 \)

That is we start from a point that is very close to the equilibrium/fixed point 0.

\[ a_0 = 0.1 \]
\[ a_1 = 2a_0(1-a_0) = 2 \cdot (0.1) \cdot (0.9) = 0.18 \]
\[ a_2 = 2a_1(1-a_1) = 2 \cdot (0.18) \cdot (0.82) = 0.2952 \]
\[ a_3 = 2a_2(1-a_2) = 2 \cdot (0.2952) \cdot (0.7048) = 0.4161 \]
\[ a_4 = ... = 0.486 \]

Hence these values suggest \( \lim_{n \to \infty} a_n = 0.5 \)

despite the fact that we started very close to 0.