Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator that has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,

- clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may receive NO credit).

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Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

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1. (a) Find a value for $c$ that makes the function

$$g(x) = \begin{cases} \frac{\sin(\pi x)}{2x} & \text{if } x \neq 0, \\ c & \text{if } x = 0 \end{cases}$$

continuous everywhere.

The function $g(x)$ is always continuous for $x \neq 0$ as it is the quotient of two continuous functions. We need to worry about $x = 0$. We need to ask that

$$C = g(0) = \lim_{x \to 0} g(x).$$

But $\lim_{x \to 0} \frac{\sin(\pi x)}{2x} = \lim_{x \to 0} \frac{\pi x}{2} \cdot \frac{\sin(\pi x)}{\pi x} = \frac{\pi}{2} \cdot \lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$

($\pi x \to 0$ as well)

Thus $C = \frac{\pi}{2}$

(b) Use the Sandwich Theorem to evaluate $\lim_{x \to 0} x^4 \cos \left(\frac{1}{x}\right)$.

Observe that

$$-x^4 \leq x^4 \cos \left(\frac{1}{x}\right) \leq x^4$$

and $\lim_{x \to 0} (-x^4) = 0 = \lim_{x \to 0} x^4$

Thus $\lim_{x \to 0} x^4 \cos \left(\frac{1}{x}\right) = 0$

by the Sandwich (or Squeeze) Theorem.
(c) Use L'Hospital's rule to evaluate \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \).

We need to apply L'Hospital's rule twice.

\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}
\]

(d) Use L'Hospital's rule to evaluate \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \).

If we use L'Hospital's rule we get

\[
\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = \lim_{x \to \infty} \frac{0}{\sqrt{x}} = 0
\]

pts: /20
2. (a) Find the equation of the tangent line to the graph of the function

\[ f(x) = \frac{xe^x}{1 + x^2} \]

at the point \( x = 0 \).

Observe that \( f(0) = \frac{0 \cdot e^0}{1 + 0^2} = 0 \).

Thus \( P(0, 0) \). Moreover,

\[ f'(x) = \frac{[1 \cdot e^x + x \cdot e^x](1 + x^2) - xe^x(2x)}{(1 + x^2)^2} = \]

\[ = \frac{e^x((1 + x)(1 + x^2) - 2x^2)}{(1 + x^2)^2} = \frac{e^x(1 + x^2 + x + x^3 - 2x^2)}{(1 + x^2)^2} \]

\[ = \frac{e^x(x^3 - x^2 + x + 1)}{(1 + x^2)^2} \]

\[ \therefore y - 0 = 1 (x - 0) \Rightarrow y = x \]

(b) Find a point on the curve

\[ y = x^2 + x + 1 \]

whose tangent is parallel to the line \( y - 2 = 3(x - 1) \).

\[ y' = 2x + 1 \]

Want to find \( x \) s.t. \( y' = 3 \)

\[ 2x + 1 = 3 \]

\[ \Rightarrow 2x = 2 \]

\[ \Rightarrow x = 1 \]

The point on the curve is therefore \( P(1, 3) \) as \( y(1) = 1^2 + 1 + 1 = 3 \)
(c) Find the interval(s) where the function

\[ g(x) = xe^{\frac{1}{2}(1-x^3)} \]

is increasing and decreasing. It is given that \[ g'(x) = \left(1 - \frac{3}{2}x^3\right)e^{\frac{1}{2}(1-x^3)}. \]

\[ g'(x) \text{ is increasing when } g'(x) > 0 \]
\[ g'(x) \text{ is decreasing when } g'(x) < 0 \]

Observe that \( e^{\frac{1}{2}(1-x^3)} \) is always positive so the sign of \( g'(x) \) is given by \( 1 - \frac{3}{2}x^3 \)

\[ g'(x) = 0 \quad \rightarrow \quad 1 - \frac{3}{2}x^3 = 0 \quad \text{or} \quad x^3 = \frac{2}{3} \]
\[ \text{or} 
\begin{align*}
  x &= \sqrt[3]{\frac{2}{3}} \\
  \downarrow &
\end{align*}
\]

Sign of \( g'(x) \)

\[ ++++++ - - - - - - - \]

(d) Suppose \( h(x) = f(x) \cdot e^{g(x)} \). Compute \( h'(2) \) given that

\[ f(2) = 4 \quad f'(2) = 7 \quad g(2) = 0 \quad g'(2) = 3. \]

\[ h'(x) = f'(x) \cdot e^{g(x)} + f(x) \cdot e^{g(x)} \cdot g'(x) \]

\[ h'(2) = f'(2) \cdot e^{g(2)} + f(2) \cdot g'(2) \cdot e^{g(2)} \]

\[ = 7 e^{0} + 4 \cdot 3 \cdot e^{0} \]

\[ = 7 + 12 \]

\[ = 19 \]

\[ \text{pts: } /20 \]
\[ x_{t+1} = f(x_t) \quad \text{where} \quad f(x) = x e^{\frac{1}{2} \left( 1 - x^3 \right)} \]

\[ f'(x) = \left( 1 - \frac{3}{2} x^3 \right) e^{\frac{1}{2} \left( 1 - x^3 \right)} \]

3. Find all the fixed points for the recursive sequence \( x_{t+1} = x_t e^{\frac{1}{2} \left( 1 - x_t^3 \right)} \).

What does the Stability Criterion say about the fixed (equilibrium) points?

To find the fixed points, we solve:
\[ x = f(x) \]

Thus:
\[ x = x e^{\frac{1}{2} \left( 1 - x^3 \right)} \quad \text{or} \quad x \left( 1 - e^{\frac{1}{2} \left( 1 - x^3 \right)} \right) = 0 \]

Thus:
\[ x = 0 \quad \text{or} \quad 1 = e^{\frac{1}{2} \left( 1 - x^3 \right)} \quad \text{or} \quad 0 = \frac{1}{2} \left( 1 - x^3 \right) \quad \text{or} \quad x = 1 \]

\[ |f'(0)| = \left| 1 - e^{\frac{1}{2}} \right| \approx 1.6487 > 1 \quad \text{so} \quad x^* = 0 \quad \text{is unstable} \]

\[ |f'(1)| = \left| 1 - \frac{3}{2} \right| \cdot e^0 = -\frac{1}{2} < 1 \quad \text{so} \quad x^* = 1 \quad \text{is locally stable} \]

Sketch a cobweb graph starting at \( x_0 = 0.5 \) and \( x_0 = 1.5 \), respectively, on each of the figures below. Use it to determine \( \lim_{t \to \infty} x_t \) in each case.

In both cases, \( \lim_{t \to \infty} x_t = 1 \)

pts: /10
4. (a) Use a logarithmic transformation to find a linear relationship between \( x \) and \( y \) if
\[
y = 3x^{-7}.
\]
\[
\log y = \log(3x^{-7}) = \log 3 + (-7) \log x
\]

or
\[
Y = -7X + \log 3
\]

when \( X = \log x \)
\[
Y = \log y
\]

(b) Given the semilog plot below, find a functional relationship between \( x \) and \( y \).

When \( x = 0 \), then \( y = 8 \); and when \( x = 3 \), then \( y = 1,000 \).

Since in a semi-log plot a line corresponds to a relation of the form \( y = a \cdot b^x \), we have:

\[
8 = a \cdot b^0 \Rightarrow a = 8
\]

\[
1000 = 8 \cdot b^3 \Rightarrow b^3 = \frac{1000}{8} \Rightarrow b = \sqrt[3]{\frac{1000}{8}} = \frac{10}{2} = 5
\]

\[
\therefore y = 8 \cdot 5^x
\]

pts: /10
5. Find the linearization of \( f(x) = \sqrt{x} \) at \( a = 8 \), and use it to approximate \( \sqrt{9} \).

\[
(a, f(a)) = (8, 2) \quad f'(x) = \frac{1}{3} x \quad f'(8) = \frac{1}{3(\sqrt{8})^2} = \frac{1}{12}
\]

\[
L(x) = 2 + \frac{1}{12}(x-8)
\]

\[
: \quad f(9) \approx L(9) = 2 + \frac{1}{12}(9-8) = 2 + \frac{1}{12} = \frac{25}{12}
\]

Observe that \( \frac{25}{12} \approx 2.083 \), \( 3\sqrt{9} \approx 2.080 \) pts: /5

6. The volumetric flow rate \( Q \) (the volume of fluid passing through a given surface per unit time) of blood moving through a cylindrical blood vessel with radius \( r \) is given by the Hagen-Poiseuille equation

\[
Q = \frac{\pi r^4}{8\mu L} \Delta p,
\]

where \( \mu \) denotes the blood viscosity, \( L \) denotes the length of the vessel, and \( \Delta p \) denotes the change in pressure along the vessel.

Assume that \( \mu, L \) and \( \Delta p \) are held constant, and that the radius is decreasing at a constant rate of \(-0.1 \) mm/s. Find the rate at which \( Q \) is decreasing when \( r = 2 \) mm.

\[
\frac{dQ}{dt} = \frac{\pi \Delta p}{8\mu L} 4r^3 \frac{dr}{dt}
\]

Thus \[
\frac{dQ}{dt} = \frac{\pi \Delta p}{2\mu L} r^3 \frac{dr}{dt}
\]

or \[
\frac{dQ}{dt} = -0.4 \frac{\pi \Delta p}{\mu L} \text{ mm}^4 \text{ mm}^{-1}
\]

pts: /5
7. (a) Compute the indefinite integral \( \int \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right) \, dx \).

\[
\int \left( x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) \, dx = \\
\frac{1}{1+\frac{1}{2}} x^{\frac{1}{2}+1} + \frac{1}{1-\frac{1}{2}} x^{-\frac{1}{2}+1} + C \\
= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C \\
= \frac{2}{3} \sqrt{x^3} + 2\sqrt{x} + C = 2\sqrt{x} \left( \frac{x}{3} + 2 \right) + C
\]

(b) Solve the initial value problem 
\[
\frac{dN}{dt} = \frac{1}{t}
\]
for \( t \geq 1 \) and \( N(1) = 10 \).

\[
N(t) = \ln(t) + c
\]

As \( N(1) = 10 \) we get
\[
10 = N(1) = \ln(1) + c \\
\therefore c = 10
\]

\[
\therefore N(t) = \ln(t) + 10
\]

pts: /10
8. (a) Evaluate \( \int_{1}^{e} \frac{x^2 + x + 1}{x} \, dx \).

\[ \int_{1}^{e} \left( x + 1 + \frac{1}{x} \right) \, dx = \left[ \frac{1}{2} x^2 + x + \ln(x) \right]_{1}^{e} \]

\[ = \left( \frac{1}{2} e^2 + e + \ln(e) \right) - \left( \frac{1}{2} (1) + 1 + \ln(1) \right) \]

\[ = \frac{1}{2} e^2 + e + 1 - \frac{1}{2} - 1 \]

\[ = \frac{1}{2} e^2 + e - \frac{1}{2} \]

(b) Use a geometric argument to evaluate \( \int_{-1}^{2} |x| \, dx \).

Area of the region shaded in the picture:

\[ = \frac{1}{2} (1) \cdot (1) + \frac{1}{2} (2) \cdot (2) \]

\[ = \frac{1}{2} + 2 = \frac{5}{2} = 2.5 \]

(c) Use a geometric argument to evaluate \( \int_{0}^{2} \sqrt{4-x^2} \, dx \).

Area of one quarter of the circle \( y^2 + x^2 = 4 \)

\[ = \frac{1}{4} \left( \pi \cdot 2^2 \right) = \pi \]

\[ = 4 \left( \frac{\pi}{4} \right) \]

\[ = \pi \]

\[ \boxed{\pi} \]

pts: /10
9. Use the Fundamental Theorem of Calculus (part 1) to evaluate the following derivatives:

(a) If \( F(x) = \int_{-1}^{x} \sqrt{u^3 + 1} \, du \), then \( F'(x) \) equals:

\[
F(x) = -\int_{-1}^{x} \sqrt{u^3 + 1} \, du \quad \text{so that}
\]

\[
F'(x) = -\sqrt{x^3 + 1} \quad \text{lim}
\]

(b) If \( G(x) = \int_{1}^{\sqrt{x}} \frac{s^2}{s^2 + 1} \, ds \), then \( \frac{dG}{dx} \) equals:

We need to use the chain rule.

Set \( u = \sqrt{x} \) then

\[
G(x) = \int_{1}^{u} \frac{s^2}{s^2 + 1} \, ds \quad \text{so} \quad \frac{d}{dx} G(x) =
\]

\[
= \frac{d}{du} \left( \int_{1}^{u} \frac{s^2}{s^2 + 1} \, ds \right) \cdot \frac{du}{dx} = \frac{u^2}{u^2 + 1} \cdot \frac{1}{2\sqrt{x}} =
\]

\[
= \frac{(\sqrt{x})^2}{(\sqrt{x})^2 + 1} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(1+x)} \quad \text{lim}
\]

pts: /10
Bonus. \(a\) (2 pts) Suppose that \(h(x)\) is a function such that
\[
    h(1) = -2 \quad h'(1) = 2 \quad h''(1) = 3 \quad h(2) = 6 \quad h'(2) = 5 \quad h''(2) = 13
\]
and \(h''(x)\) is continuous everywhere. Evaluate \(\int_1^2 h''(u) \, du\).

\[
    = h''(2) - h''(1) = 5 - 2 = 3
\]

\[
    \text{as } h'' = \text{antiderivative of } h' \text{ i.e. } h' \text{ is an antiderivative of } h''.
\]

\(b\) (3 pts) Find a function \(f\) and a number \(a\) such that
\[
    6 + \int_a^x \frac{f(t)}{t^2} \, dt = 2\sqrt{x}.
\]
If we plug in "a" we get
\[
    6 + \int_a^a \frac{f(t)}{t^2} \, dt = 6 \implies a = 9.
\]

\(c\) (5 pts) Suppose that the length of a certain organism at age \(t\) is given by \(L(t)\), which satisfies the differential equation
\[
    \frac{dL}{dt} = e^{-0.1t}, \quad t \geq 0.
\]
Find \(L(t)\) if the limiting length \(L_\infty\) is given by \(L_\infty = \lim_{t \to \infty} L(t) = 25\).

How long is the organism at age \(t = 0\)?

\[
    \frac{dL}{dt} = e^{-0.1t} \implies L(t) = \frac{1}{-0.1} e^{-0.1t} + C \implies L(t) = -10 e^{-0.1t} + C
\]

\[
    \lim_{t \to \infty} (-10 e^{-0.1t} + C) = 0 + C = 25 \implies L(t) = 25 - 10 e^{-0.1t}
\]

\[
    L(0) = 25 - 10 e^0 = 25 - 10 = 15
\]

\[
    \boxed{\text{pts: } /10}
\]