1. Consider the function defined implicitly near (2, 1) by \(x^2 y^2 + 3xy = 10y\).
   (a) [7 points] Use implicit differentiation to find the derivative at (2, 1).

\[
\frac{d}{dx} [x^2 y^2 + 3xy] = \frac{d}{dx} [10y]
\]
\[
2xy^2 + 2y^2 \frac{dy}{dx} + 3y + 3x \frac{dy}{dx} = 10 \frac{dy}{dx}
\]
\[
2y^2 \frac{dy}{dx} + 3x \frac{dy}{dx} - 10 \frac{dy}{dx} = -2xy^2 - 3y
\]
\[
\frac{dy}{dx} = \frac{-2xy^2 + 3y}{2y^2 + 3x - 10}
\]

Substituting \(x = 2\) and \(y = 1\), we get
\[
\frac{dy}{dx} \bigg|_{(2,1)} = \frac{4 + 3}{8 + 6 - 10} = \frac{7}{4}
\]

(b) [3 points] Determine the equation of the tangent line at (2, 1).

The equation of the tangent line is \(y = -\frac{7}{4}(x - 2) + 1\).

2. Let \(f(x)\) be the function defined by \(f(x) = e^{\sqrt{x}}\) for \(x \geq 0\).
   (a) [6 points] Write an equation for the line tangent to the graph of \(f(x)\) at the point where \(x = 1\).

\[
f'(x) = \frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}
\]

So the slope of the tangent line at \(x = 1\) is \(f'(1) = \frac{e}{2}\). The equation for the tangent line is \(y = \frac{e}{2}(x - 1) + e\).
(b) [4 points] Find the point where this tangent line crosses the x-axis.

The line crosses the x-axis when $y = 0$.

\[
\frac{e}{2}(x - 1) + e = 0 \\
\frac{e}{2}(x - 1) = -e \\
x - 1 = -2 \\
x = -1
\]

The tangent line crosses the x-axis at $(-1, 0)$.

3. Let $f$ be the function defined by $f(x) = x^3 - x^2 - 2x$ for $0 \leq x \leq 3$.

(a) [2 points] Find where the graph of $f$ crosses the x-axis. Show your work.

Set $f(x) = 0$ and solve

\[
x^3 - x^2 - 2x = 0 \\
x(x^2 - x - 2) = 0 \\
x(x + 1)(x - 2) = 0
\]

The solutions are $x = -1$, $x = 0$, $x = 2$, but we are restricted to $[0, 3]$, so the only two places where $f$ crosses the x-axis will be $x = 0$ and $x = 2$.

(b) [4 points] Find the intervals on which $f$ is increasing.

Find the derivative and the critical points.

\[f'(x) = 3x^2 - 2x - 2\]

Now solve

\[
3x^2 - 2x - 2 = 0 \\
x = \frac{2 \pm \sqrt{28}}{6} = \frac{1 \pm \sqrt{7}}{3}
\]

Only $x = \frac{1 + \sqrt{7}}{3}$ lies in our interval. We know that $f(x)$ is decreasing if $f'(x) < 0$ which happens when $0 < x < \frac{1 + \sqrt{7}}{3}$. Also, $f(x)$ is increasing if $f'(x) > 0$ which happens when $\frac{1 + \sqrt{7}}{3} < x < 3$.

(c) [4 points] Find the absolute maximum and absolute minimum value of $f$. Justify your answer.

$f(x)$ is continuous on the closed interval $[0, 3]$, so it attains its maximum and minimum. There are three points to check: $x = 0$, $x = \frac{1 + \sqrt{7}}{3}$, and $x = 3$. 
From the First Derivative Test we can see that the absolute minimum occurs at \( x = \frac{1 + \sqrt{7}}{3} \). Evaluating \( f \) at 0 and 3, we have that \( f(0) = 0 \) and \( f(3) = 12 \), so the absolute maximum occurs at \( x = 3 \).

The maximum value of \( f \) on \([0, 3]\) is 12 and the minimum value is \(-\frac{20 + 14\sqrt{7}}{27} \approx -2.1126\).

4. **[10 points]** A projectile is fired toward the sea from a cliff 200 feet above the water. If the launch angle is 45\(^\circ\) and the muzzle velocity is \( k \), then the height \( h \) of the projectile above the sea when it is \( x \) feet downrange from the launch site is

\[
h(x) = -\frac{32}{k^2} x^2 + x + 200 \text{ feet.}
\]

See the figure.

Determine the maximum values of \( h \) and the distance downrange where this maximum occurs if \( k = 2800 \) feet per second.

\[
h(x) = -\frac{32}{k^2} x^2 + x + 200
\]
\[
h'(x) = -\frac{64}{k^2} x + 1
\]

Set equal to zero and solve

\[-\frac{64}{k^2} x + 1 = 0
\]
\[x = \frac{k^2}{64}
\]

Since \( h''(x) < 0 \), this will be the absolute maximum. Thus, the downrange distance is \( x = \frac{k^2}{64} = 122,500 \) feet. The height at that distance is \( h(122500) = 61,450 \) feet.
5. Find the requested quantities in each of the following.

(a) [5 points] Let

\[ f(x) = \begin{cases} 
2 + \cos(x - 1) & \text{if } 0 \leq x \leq 1, \\
ax - 2 & \text{if } 1 < x < 2, \\
x^2 - 2x + b & \text{if } 2 \leq x \leq 3.
\]

where \(a\) and \(b\) are constants. Find the values of \(a\) and \(b\) which make \(f(x)\) is continuous on \([0, 3]\).

We need \(\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)\). This gives the equation

\[ 2 + \cos(0) = a - 2 \]
\[ 2 + 1 = a - 2 \]
\[ 3 = a - 2 \]
\[ a = 5 \]

Likewise, we need \(\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)\). This gives

\[ 2a - 2 = 2^2 - 2(2) + b \]
\[ 2a - 2 = 4 - 4 + b \]
\[ 2a - 2 = b \]
\[ 2(5) - 2 = b \]
\[ 10 - 2 = b \]
\[ b = 8 \]

These values for \(a\) and \(b\) will make \(f(x)\) continuous.

(b) [5 points] Evaluate \(\lim_{n \to \infty} \frac{6n^2 - 3n + 7}{(n + 3)^2}\).

\[
\lim_{n \to \infty} \frac{6n^2 - 3n + 7}{(n + 3)^2} = \lim_{n \to \infty} \frac{6n^2 - 3n + 7}{n^2 + 6n + 9} = \lim_{n \to \infty} \frac{6n^2/n^2 - 3n/n^2 + 7/n^2}{n^2/n^2 + 6n/n^2 + 9/n^2} = \frac{6}{1} = 6
\]

6. Find the following limits

(a) [4 points] \(\lim_{x \to 0} \frac{\sin(x)}{x \cos(x)}\).

Use l’Hospital’s Rule since both the numerator and denominator go to 0.

\[
\lim_{x \to 0} \frac{\sin(x)}{x \cos(x)} = \lim_{x \to 0} \frac{\cos(x)}{\cos(x) - x \sin(x)} = \frac{1}{1 - 0} = 1
\]
This can also be done as follows:

\[
\lim_{x \to 0} \frac{\sin(x)}{x \cos(x)} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1 \cdot 1 = 1
\]

(b) [4 points] \( \lim_{x \to \infty} \frac{\ln(xe^x)}{x} \).

Use l’Hospital’s Rule since both the numerator and denominator go to infinity.

\[
\lim_{x \to \infty} \frac{\ln(xe^x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{xe^x} (e^x + xe^x)}{1} = \lim_{x \to \infty} \frac{e^x + xe^x}{xe^x} = \lim_{x \to \infty} \frac{1}{x} + 1 = 1
\]

(c) [5 points] \( \lim_{x \to 2} \frac{x^2 - 4}{x^2 - x + 6} \).

l’Hospital’s Rule does not apply.

\[
\lim_{x \to 2} \frac{x^2 - 4}{x^2 - x + 6} = \frac{0}{8} = 0
\]

7. Find the following general antiderivatives.

(a) [4 points] \( \int (x^2 + x + x^{-1} + x^{-2}) \, dx \)

\[
\int (x^2 + x + x^{-1} + x^{-2}) \, dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \ln|x| - \frac{1}{x} + C.
\]

(b) [3 points] \( \int (2\sin(x) - 2\cos(x) + 2\sec^2(x)) \, dx \)

\[
\int (2\sin(x) - 2\cos(x) + 2\sec^2(x)) \, dx = -2\cos(x) - 2\sin(x) + 2\tan(x) + C.
\]

(c) [3 points] \( \int \left( \frac{e^x - e^{-x}}{2} \right) \, dx \).

\[
\int \left( \frac{e^x - e^{-x}}{2} \right) \, dx = \frac{1}{2} \int (e^x - e^{-x}) \, dx = \frac{1}{2}(e^x + e^{-x}) + C
\]
8. Use the Fundamental Theorem of Calculus (part 1) to evaluate the following derivatives.

(a) [5 points] If \( F(x) = \int_1^x \cos^2(u)e^{-u^2} \, du \) then \( F'(x) \) equals

By the Fundamental Theorem of Calculus, Part I, we know that

\[
F'(x) = \frac{d}{dx} \int_1^x \cos^2(u)e^{-u^2} \, du = \cos^2(x)e^{-x^2}.
\]

(b) [5 points] If \( G(x) = \int_1^{3x^2} \cos(2u)\sin(2u) \, du \) then \( G'(x) \) equals

By the Fundamental Theorem of Calculus, Part I, and the Chain Rule we have

\[
G'(x) = \frac{d}{dx} \int_1^{3x^2} \cos(2u)\sin(2u) \, du
= \cos\left(2(3x^2)\right)\sin\left(2(3x^2)\right) \frac{d}{dx}(3x^2)
= 6x\cos(6x^2)\sin(6x^2)
\]

9. Find the following definite integrals.

(a) [5 points] \( \int_1^3 4x - \frac{1}{x} \, dx \)

\[
\int_1^3 4x - \frac{1}{x} \, dx = 2x^2 - \ln(x) \bigg|_1^3
= (18 - \ln 3) - (2 - \ln 1)
= 16 - \ln 3 \approx 14.9014
\]

(They may use the calculator to compute this integral, but in that case the answer is either right or wrong — no partial credit.)
(b) [5 points] The function $h(x)$ is graphed below.

Compute $\int_0^{10} h(x) \, dx$.
This is to be done geometrically.

$$
\int_0^{10} h(x) \, dx = \int_0^1 h(x) \, dx + \int_1^2 h(x) \, dx + \int_2^4 h(x) \, dx + \int_4^5 h(x) \, dx + \int_5^{10} h(x) \, dx
$$

$$
= (0)(1) + \frac{1}{2}(2 + 4)(1) + \frac{1}{2}(4 + 2)(2) + (2)(1) + \frac{1}{2}(1 + 5)(5)
$$

$$
= 26
$$

10. Find the requested areas.

(a) [3 points] Set up (but do NOT evaluate) the integral that will give the area enclosed by the curves $y = \cos(x)$, $y = 1 + \sin(2x)$ and the vertical line $x = \frac{3\pi}{2}$.

Area = $\int_0^{\frac{3\pi}{2}} [(1 + \sin(x)) - \cos(x)] \, dx$

(The correct answer is $2 + \frac{3\pi}{2} \approx 6.71239$. They do not have to evaluate the integral, but IF they do, they must do it correctly.)
(b) [7 points] The region \( R \) is enclosed by \( y = x/4 \) and \( y = \sqrt{x} \) in the first quadrant. **Find** the area of \( R \).

First, find the points of intersection by setting \( \sqrt{x} = \frac{x}{4} \). These are \( x = 0 \) and \( x = 16 \). The area is then

\[
\text{Area}_R = \int_0^{16} \left( \sqrt{x} - \frac{x}{4} \right) dx = \frac{32}{3} \approx 10.6667.
\]

(They may use the calculator to compute this integral, but they must show the integral that they are evaluating and not just the answer.)

**BONUS [2 points each]** Compute the derivatives of the following functions. You do not need to simplify.

(a) \( f(x) = \frac{x^2 + 1}{e^{2x}} \).

\[
f'(x) = \frac{2xe^{2x} - 2(x^2 + 1)e^{2x}}{e^{4x}}
\]

(b) \( f(x) = x^5 \cdot \sin(2x) \).

\[
f'(x) = 5x^4 \sin(2x) + 2x^5 \cos(2x).
\]

(c) \( f(x) = \cos(3x + 5x^3) \).

\[
f'(x) = -(3 + 15x^2) \sin(3x + 5x^3).
\]

(d) \( f(x) = e^x \).

\[
f'(x) = 0
\]

(e) \( f(x) = (4x^2 - 1)^3 + (8x + 9)^{11} \).

\[
f'(x) = 3(4x^2 - 1)^2(8x) + 11(8x + 9)^{10}(8).
\]