Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator that has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may receive NO credit).

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Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

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1. Match each of the following direction fields (or, better, phase portraits), labeled A-C, with the appropriate differential equation, labeled I-III:

Answers:
A: III
B: II
C: I

pts: /10
Notice that \( \frac{dy}{dx} = -\frac{x}{y} \) corresponds to

\[
\int y \, dy = \int -x \, dx \quad \Rightarrow \quad \frac{1}{2} y^2 = -\frac{1}{2} x^2 + C
\]

That is, the general solution is of the form

\[
x^2 + y^2 = 2C
\]

These are concentric circles centered at the origin (for \( C > 0 \))

\[\int\]

\( \frac{dy}{dx} = x(2-y) \) has general solution

\[
y = 2 - B \, e^{-\frac{1}{2} x^2}
\]

as shown in problem #2.

\[\int\]

Similar to problem #3.

\[
\frac{dy}{dx} = 2-y \quad \Leftrightarrow \quad \frac{-1}{2-y} \, dy = - \, dx
\]

\[
\ln(2-y) = -x + C \quad \text{or} \quad 2-y = A \, e^{-x}
\]

\[
i \quad y = 2 - A \, e^{-x} \quad (A = e^C)
\]
2. Solve the differential equation

\[
\frac{dy}{dx} + xy = 2x
\]

with initial condition \( y(0) = 1 \).

\[
\frac{dy}{dx} = 2x - xy
\]

\[
\frac{dy}{dx} = x(2-y)
\]

Separate the variables:

\[
\frac{dy}{2-y} = x \, dx
\]

and rewrite as

\[
\int \frac{-1}{2-y} \, dy = \int -x \, dx
\]

so we get

\[
\ln |2-y| = -\frac{1}{2} x^2 + C
\]

Take the exponential to get

\[
|2-y| = A \cdot e^{-\frac{1}{2} x^2}
\]

where \( A = e^C \).

Now \( 2-y = \pm A e^{-\frac{1}{2} x^2} \) or

\[
y = 2 - B e^{-\frac{1}{2} x^2}
\]

The initial condition \( y(0) = 1 \) gives

\[
y = 2 - B e^0 \quad \therefore B = 1 \quad \text{so} \quad \boxed{y = 2 - e^{-\frac{1}{2} x^2}}
\]

pts: /10
Differential equations have been used extensively in the study of drug dissolution for patients given oral medications. One of the simplest such models is given by the Noyes-Whitney equation.

3. The dynamics of the drug concentration $c$ (in mg/mL) in the Noyes-Whitney model is governed by the differential equation

$$\frac{dc}{dt} = k(c_s - c),$$

where $k > 0$ and $c_s > 0$ are constants: $k$ is the dissolution rate of the drug and $c_s$ is the drug concentration in the stagnant layer.

(i) (2 pts) Is this differential equation pure-time, autonomous, or something else?

It is autonomous

(ii) (6 pts) Solve the differential equation for the initial condition $c(0) = 0$.

$$\frac{dc}{c_s - c} = k \, dt \quad \text{or} \quad \int \frac{dc}{c - c_s} = \int k \, dt$$

Thus, $\ln(c - c_s) = -kt + C$. Take the exponential of both sides:

$$c - c_s = A e^{-kt}$$

When $A = e^C$, we get

$$c - c_s = A e^{c_s}$$

As $c(0) = 0$, we get $0 = c_s = A e^{c_s}$.

As $c(0) = 0$, we get

$$c(t) = c_s (1 - e^{-kt})$$

(iii) (2 pts) Using (ii), find the limit of the drug concentration $c$ as $t \to \infty$.

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} c_s (1 - e^{-kt}) = c_s$$

as $e^{-kt} \to 0$ as $t \to \infty$

**pts:** /10
4. **A simple model of predation:** Suppose that $N(t)$ denotes the size of a population at time $t$. The population evolves according to the logistic equation, but, in addition, predation reduces the size of the population so that the rate of change is given by

$$\frac{dN}{dt} = N \left(1 - \frac{N}{20}\right) - \frac{7N}{4+N}. \quad (*)$$

The first term on the right-hand side describes the logistic growth; the second term describes the effect of predation.

For your convenience, the function $g(N) = N \left(1 - \frac{N}{20}\right) - \frac{7N}{4+N}$ is graphed below.

(a) (4 pts) Find (algebraically) all the equilibria $\hat{N}$ of $(*)$.

We need to set $g(N) = 0$

$$N \left(1 - \frac{N}{20}\right) - \frac{7N}{4+N} = 0 \quad \iff \quad N \left[\frac{20-N}{20} - \frac{7}{4+N}\right] = 0 \quad \iff \quad N \left[\frac{20-N}{20} - \frac{7}{4+N}\right] = 0 \quad \iff \quad N \left[(20-N)(4+N) - 7 \cdot 20\right] = 0 \quad \iff \quad \frac{N\left[80+16N-N^2-140\right]}{20(4+N)} = 0 \quad \iff \quad -N\left(N^2-16N+60\right) = 0 \quad \iff \quad N(N-6)(N-10) = 0$$
(b) (4 pts) Use the graph of \( g(N) \) to classify the stability of the equilibria \( \hat{N} \) found in (a).

\[ \hat{N} = 0 \text{ is locally stable} \]
\[ \hat{N} = 10 \text{ is locally stable} \]
\[ \hat{N} = 6 \text{ is unstable} \]

(c) (4 pts) Find \( g'(N) \), where \( g(N) = N \left(1 - \frac{N}{20}\right) - \frac{7N}{4+N} \).

\[ g(N) = N - \frac{N^2}{20} - \frac{7N}{4+N} \]

\[ g'(N) = 1 - \frac{N}{10} - \frac{7(N+4) - 7N(1)}{(4+N)^2} \]

\[ g'(N) = 1 - \frac{N}{10} - \frac{28}{(4+N)^2} \]

(d) (3 pts) Use the eigenvalues method (stability criterion) to classify the stability of the equilibria \( \hat{N} \) found in (a).

\[ g'(0) = 1 - \frac{28}{4^2} = -0.75 < 0 \]
\[ \hat{N} = 0 \text{ locally stable} \]

\[ g'(6) = 1 - \frac{6}{10} - \frac{28}{10^2} = 0.12 > 0 \]
\[ \hat{N} = 6 \text{ unstable} \]

\[ g'(10) = 1 - \frac{10}{10} - \frac{28}{(14)^2} = -0.143 < 0 \]
\[ \hat{N} = 10 \text{ locally stable} \]
5. Consider the autonomous differential equation \( \frac{dy}{dx} = g(y) \).

The graph of \( dy/dx \) as a function of \( y \) is given below.

(a) (2 pts) Use the graph on the right to find the equilibria \( \hat{y} \) of the differential equation.

\[ \hat{y} = -4, \quad 0, \quad 4 \]

(b) (5 pts) Use the graph on the right and the geometric approach to discuss the stability of the equilibria you found in (a).

For values of \( y \) smaller than \(-4\), \( \frac{dy}{dx} > 0 \). So \( y(x) \) is increasing, so \( y \to -4 \). Similarly for values larger than \(-4\), \( \frac{dy}{dx} < 0 \). So \( y(x) \) is decreasing.

(c) (3 pts) Using the information found in (a) and (b), which of the following direction fields (phase portraits) matches the given differential equation? Circle the correct one.
6. (9 pts) Solve the following system of linear equations

\[
\begin{align*}
\begin{cases}
x + 2y - z &= 2 \\
x + 4y + 3z &= 8 \\
3x + 8y + z &= 12
\end{cases}
\end{align*}
\]

by writing the corresponding augmented matrix and then by row reducing.

\[
\begin{bmatrix}
1 & 2 & -1 & 2 \\
1 & 4 & 3 & 8 \\
3 & 8 & 1 & 12
\end{bmatrix} \rightarrow R_2 - R_1 \begin{bmatrix}
0 & 2 & 4 & 6 \\
0 & 2 & 4 & 6
\end{bmatrix}
\]

\[
R_3 - R_2 \begin{bmatrix}
1 & 2 & -1 & 2 \\
0 & 2 & 4 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix} \rightarrow \frac{1}{2} R_2 \begin{bmatrix}
1 & 2 & -1 & 2 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
R_1 - 2R_2 \begin{bmatrix}
1 & 0 & -5 & -4 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Rightarrow \begin{cases}
x_1 - 5x_3 = -4 \\
x_2 + 2x_3 = 3
\end{cases}
\]

\[
\begin{cases}
x_1 = -4 + 5x_3 \\
x_2 = 3 - 2x_3
\end{cases}
\]

\[x_3 \text{ any value } t \in \mathbb{R}\]

So

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-4 \\
3 \\
0
\end{bmatrix} + t \begin{bmatrix}
5 \\
-2 \\
1
\end{bmatrix} \quad t \in \mathbb{R}
\]
(3 pts) How many solutions does the system have?

The system has infinitely many solutions.

(3 pts) Which of the two pictures below illustrates the geometric situation described by the given system of linear equations? Explain your choice.
7. (a) (5 pts) The solution(s) for the system of linear equations corresponding to the following augmented matrix

\[
\begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & -7 & 10 \\
0 & 0 & 0 & -5 \\
\end{bmatrix}
\]

is (are):

This system has no solution as the third row says \( 0 = -5 \), which is impossible.

The system is inconsistent.

(b) (5 pts) The solution(s) for the system of linear equations corresponding to the following augmented matrix

\[
\begin{bmatrix}
1 & 0 & 4 & 6 \\
0 & 1 & -5 & -4 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

is (are):

This augmented matrix means

\[
\begin{align*}
& x_1 + 4x_3 = 6 \\
& x_2 - 5x_3 = -4
\end{align*}
\]

or

\[
\begin{cases}
& x_1 = 6 - 4x_3 \\
& x_2 = -4 + 5x_3
\end{cases}
\]

Thus \( x_3 \) can take on any value \( t \in \mathbb{R} \) and

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}
\]

in matrix form.

pts: /10
8. Let \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 \\ 0 & 8 \\ -6 & 14 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \).

(a) Compute \( 3A - \frac{1}{2}B \).

\[
3 \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 4 & -2 \\ 0 & 8 \\ -6 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & -1 \\ 6 & -7 \end{bmatrix}
\]

(b) Compute the product \( AC \).

\( A \) is \( 3 \times 2 \) and \( C \) is a \( 2 \times 3 \) matrix. Thus, \( AC \) is a \( 3 \times 3 \) matrix:

\[
AC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}
\]

(c) Compute the product \( CA \).

\( CA \) is a \( 2 \times 2 \) matrix:

\[
CA = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix}
\]

pts: /10
9. Find the values of $a$ and $b$ that satisfy the following matrix equation:
\[
\begin{bmatrix}
0 & 2 \\
6-a & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 2a \\
1 & 2
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
-40 & 8 \\
2 & b
\end{bmatrix}.
\]

(Recall that the notation $A^T$ denotes the transpose of the matrix $A$. Our textbook uses the alternative notation $A'$.)

\[
\begin{bmatrix}
0 & 2 \\
6-a & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 1 \\
2a & 2
\end{bmatrix} = \begin{bmatrix}
-20 & 4 \\
1 & b/2
\end{bmatrix}
\]

\[
\begin{bmatrix}
4a & 4 \\
(6-a)+2a & 6-a+2
\end{bmatrix} = \begin{bmatrix}
-20 & 4 \\
1 & b/2
\end{bmatrix}
\]

\[
\begin{bmatrix}
4a & 4 \\
6+a & 8-a
\end{bmatrix} = \begin{bmatrix}
-20 & 4 \\
1 & b/2
\end{bmatrix}
\]

Two matrices are equal if and only if the corresponding entries are equal.

So
\[
4a = -20 \quad 4 = 4 \quad \checkmark
\]
\[
a = -5 \quad \checkmark
\]
\[
6 + a = 1 \quad 8 - a = b/2 \quad \Rightarrow b = 26
\]

\[\text{pts: } /10\]
Bonus. Choose only one of the following two problems:

(a) A pharmaceutical company makes up a vitamin capsule using different recipes for men and women. In each capsule, the recipe calls for 80 units of vitamin B\textsubscript{12} and 50 units of vitamin C in the capsules for men, and 70 units of vitamin B\textsubscript{12} and 40 units of vitamin C in the capsules for women.

If the manufacturer uses 13 units of vitamin C for every 22 units of vitamin B\textsubscript{12}, then what is the gender ratio of the customer base?

(b) There are various possibilities for modeling tumor growth.
For instance, a tumor can be modeled as a spherical collection of cells and that it acquires resources for growth only through its surface area. All cells in a tumor are also subject to a constant per capita death rate. With these assumptions, the dynamics of tumor mass $M$ (in grams) is therefore modeled by the differential equation

$$\frac{dM}{dt} = \kappa M^{2/3} - \mu M,$$

where $\kappa$ and $\mu$ are positive constants. The first term represents tumor growth via nutrients entering through the surface; the second term represents a constant per capita death rate.

Suppose $\kappa = 1$, that is the dynamics of tumor mass is modeled as

$$\frac{dM}{dt} = M^{2/3} - \mu M.$$

Which value does the tumor mass approach as time $t \to \infty$? Justify your answer.

\[\begin{align*}
\text{(a)} \quad & x = \text{number of men} \\
& y = \text{number of women} \\
& \text{If } 22u \text{ units of vitamin } B_{12} \text{ are used, then } 13u \text{ are the units of vitamin } C \text{ used} \\
& \begin{cases}
80x + 70y = 22u \\
50x + 40y = 13u
\end{cases} \\
\text{The augmented system is:} \\
\begin{bmatrix}
8 & 7 & 2.2u \\
5 & 4 & 1.3u
\end{bmatrix} \\
\text{pts: } /10
\]
\[
\begin{bmatrix}
1 & \frac{7}{8} & \frac{9.2}{8} u \\
5 & 4 & 1.3 u
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & \frac{7}{8} & \frac{1.1}{4} u \\
0 & -\frac{3}{8} & -\frac{3}{4} u
\end{bmatrix}
\]

\[-\frac{8}{3} R_2
\begin{bmatrix}
1 & \frac{7}{8} & 0.275 u \\
0 & 1 & 0.2 u
\end{bmatrix}
\]

\[R_1 - \frac{7}{8} R_2
\begin{bmatrix}
1 & 0 & 0.1 u \\
0 & 1 & 0.2 u
\end{bmatrix}
\]

This means that \( x = 0.1 u \) and \( y = 0.2 u \)

The ratio is \( \frac{x}{y} = \frac{0.1 u}{0.2 u} \) or \( x = \frac{1}{2} y \)

This means that there are twice as many women as men in the customer base.

(b) \[ \frac{dM}{dt} = M^{2/3} - \mu M \] 

the equilibria are obtained by solving \( g(M) = 0 \) or \( M^{2/3} - \mu M = 0 \)

\[ \implies M (M^{-1/3} - \mu) = 0 \] 

Thus \( \hat{M} = 0 \) or \( \hat{M} = M^{-3} = \sqrt[3]{\frac{1}{\mu^3}} \)
Using the stability criterion

\[ q'(M) = \frac{2}{3} M^{-\frac{1}{3}} - \mu \]

\[ q'(0) \text{ is not defined (it tends to } +\infty \text{) as } M \to 0^+ \]

\[ q'\left(\frac{1}{\mu^3}\right) = q'(\mu^{-3}) = \frac{2}{3} (\mu^{-3})^{-\frac{1}{3}} - \mu \]

\[ = \frac{2}{3} \mu - \mu = -\frac{1}{3} \mu < 0 \]

\[ \therefore \ \frac{1}{\mu^3} = \hat{M} \text{ is a stable equilibrium} \]

Thus the mass \( M \) of the tumor approaches \( \frac{1}{\mu^3} \) as \( t \to \infty \).

→ Notice that if the mortality rate of the tumor cell is high, then \( \frac{1}{\mu^3} \) is small.

→ If the mortality rate of the tumor cell is low, then \( \frac{1}{\mu^3} \) is high/big.