Answer all of the following questions. Use the backs of the question papers for scratch paper. No books or notes may be used. You may use a calculator. You may not use a calculator that has symbolic manipulation capabilities. When answering these questions, please be sure to:

- check answers when possible,
- clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may receive NO credit).

<table>
<thead>
<tr>
<th>QUESTION</th>
<th>SCORE</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>10</td>
<td></td>
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<td>10</td>
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<td>5.</td>
<td>15</td>
<td></td>
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<td>6.</td>
<td>10</td>
<td></td>
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<td>7.</td>
<td>15</td>
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<td>8.</td>
<td>10</td>
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<td>9.</td>
<td>10</td>
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<tr>
<td>Bonus.</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

<table>
<thead>
<tr>
<th>Sections #</th>
<th>Lecturer</th>
<th>Time/Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>001-004</td>
<td>Alberto Corso</td>
<td>MWF 08:00 am - 08:50 pm, CB 114</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section #</th>
<th>Recitation Instructor</th>
<th>Time/Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>001</td>
<td>Dustin Hedmark</td>
<td>MW 02:00 pm - 02:50 pm, CB 342</td>
</tr>
<tr>
<td>002</td>
<td>Dustin Hedmark</td>
<td>MW 03:00 pm - 03:50 pm, CB 241</td>
</tr>
<tr>
<td>003</td>
<td>Wesley Hough</td>
<td>TR 12:30 pm - 01:20 pm, DH 331</td>
</tr>
<tr>
<td>004</td>
<td>Wesley Hough</td>
<td>TR 02:00 pm - 02:50 pm, TEB 231</td>
</tr>
</tbody>
</table>
1. Verify that 

\[ y = -t \cos t - t \]

is a solution of the initial-value problem

\[ t \frac{dy}{dt} = y + t^2 \sin t \quad y(\pi) = 0. \]

\[
\frac{dy}{dt} = - t \cos t - t (- \sin t) - 1
\]

product rule

Thus if we substitute in the D.E. we get

\[ t \left[ - \cos t + t \sin t - 1 \right] \quad \frac{dy}{dx} \quad \frac{dy}{dt} \]

\[ \Rightarrow \left[ - t \cos t - t \right] + t^2 \sin t \]

\[ -t \cos t + t^2 \sin t - t \quad \Rightarrow \quad -t \cos t - t + t^2 \sin t \]

Yes

Notice that:

\[ y(\pi) = -\pi \cos(\pi) - \pi \]

\[ = -\pi (-1) - \pi = \pi - \pi = 0 \]

the initial condition is satisfied

pts: 10
2. Find the solution of the initial value problem

\[
\frac{dy}{dx} = xy^3 \quad y(0) = 2.
\]

Separate the variables

\[
\frac{1}{y^3} \, dy = x \, dx \quad \text{and integrate}
\]

\[
\int y^{-3} \, dy = \int x \, dx
\]

\[
\frac{-1}{2} y^{-2} = \frac{1}{2} x^2 + C
\]

Multiply both sides by \(-2\)

\[
y^{-2} = C - x^2 \quad \overset{\text{C}}{=} \quad C = -2C
\]

The initial condition says \(y(0) = 2\)

So \(\frac{1}{4} = C - 0 \quad \therefore \quad C = \frac{1}{4}\)

Thus \(y^{-2} = \frac{1}{4} - x^2\) or

\[
y^2 = \frac{1}{\frac{1}{4} - x^2}
\]

\[
y = \sqrt{\frac{4}{1-4x^2}}
\]

pts: /10
3. Consider the autonomous differential equation \( \frac{dy}{dx} = y(a - y)(b - y) \), where \( b > a > 0 \).

The graph of \( \frac{dy}{dx} = g(y) \) as a function of \( y \) is given below.

(1) (2 pts) Find the equilibria \( \hat{y} \) of the differential equation.

\[ \hat{y} = 0, a, b \]

(2) (5 pts) Use the graph on the right and the geometric approach to discuss the stability of the equilibria you found in (1).

(3) (3 pts) Using the information found in (1) and (2), sketch the arrows of the direction field that corresponds to the given differential equation? (Where do the arrows point?)

\[ \hat{y} = a \text{ locally stable} \]

\[ \hat{y} = b \text{ locally stable} \]

\[ \hat{y} = 0 \text{ semi-stable} \]

pts: /10
Example: $\frac{dy}{dx} = y(a-y)^2 \cdot (b-y)$

with $a = 1.25$ and $b = 3$
4. A phase line for an autonomous differential equation \( \frac{dy}{dt} = f(y) \) is shown below.

\[\begin{align*}
\text{\( y \)} & \quad \text{\(-3\)} & \quad \text{\( 0 \)} & \quad \text{\( 2 \)} \\
\end{align*}\]

Which graph A-D most closely matches the graph corresponding to the differential equation? Explain your choice.

- **A:**
- **B:**
- **C:** Correct
- **D:** Wrong equilibrium

**pts:** /10
5. (Gompertz Model of Tumor Growth) In this tumor growth model it is assumed that the per volume growth rate of the tumor declines as the tumor volume gets larger according to the equation

\[
\frac{dV}{dt} = a V \left( \ln b - \ln V \right) / \sigma(V)
\]

where \(a\) and \(b\) are positive constants.

1. (3 pts) Find the two equilibria \(\hat{V}\) of the given differential equation.

2. (5 pts) Use the stability criterion (eigenvalues method) to classify testability of the equilibria \(\hat{V}\) found in (1).

3. (7 pts) Show that the solution of this DE with initial tumor volume \(V(0) = V_0\) is

\[
V(t) = b \cdot \exp \left[ - \ln \left( \frac{b}{V_0} \right) \exp(-at) \right].
\]

and verify that \(\lim_{t \to \infty} V(t) = b\).

1. \[g(V) = 0 \quad \Rightarrow \quad \hat{V} = 0\]

2. \[g'(V) = a \left( \ln b - \ln(V) \right) + aV \cdot \left( -\frac{1}{V} \right) = a \ln b - a \ln V - a\]

If \(\hat{V} = 0\) then \(g'(0) = +\infty\)

\[
g'(b) = -a < 0
\]

\(\hat{V} = b\) locally stable [pts. /15]
\( \frac{dV}{dt} = a \sqrt{V \left( \frac{1}{u} - \frac{1}{v} \right)} \)

Separate variables

\[ \frac{1}{\sqrt{V}} \frac{dV}{\ln u - \ln v} = a \, dt \]

Note that \( \frac{d}{dt} (\ln u - \ln v) = -\frac{1}{v} \)

So we multiply both sides by \((-1)\) and then integrate

\[ \int \left( -\frac{1}{v} \right) \frac{dV}{\ln u - \ln v} = \int -a \, dt \]

\[ \ln \left[ \ln u - \ln v \right] = -at + C \]

Take the exponential of both sides

\[ \ln u - \ln v = A \cdot e^{-at} \]

Use \( V = V(0) \), we obtain

\[ A = e^C \]
\[ \ln b - \ln V_0 = A \cdot e^t \]

\[ \therefore A = \ln \left( \frac{b}{V_0} \right) \]

Thus:

\[ \ln b - \ln V = \ln \left( \frac{b}{V_0} \right) e^{-at} \]

\[ \therefore \ln V = \ln b - \ln \left( \frac{b}{V_0} \right) e^{-at} \]

Take exponential again:

\[ V = e^{\ln b - \ln \left( \frac{b}{V_0} \right) e^{-at}} \]

\[ = \left[ \frac{b}{e^{-\ln \left( \frac{b}{V_0} \right) e^{-at}}} \right] \]

\[ \lim_{t \to \infty} b \cdot e^{-\ln \left( \frac{b}{V_0} \right) e^{-at}} = \]

\[ = b \cdot e^{-\ln \left( \frac{b}{V_0} \right) \cdot \lim_{t \to \infty} e^{-at}} \]

\[ = b \cdot e^0 = b \]
6. (1) (5 pts) Find the value of $k$ for which the system is consistent

\[
\begin{cases}
-9x + 6y = 0 \\
-18x + ky = -3
\end{cases}
\]

\[
\begin{bmatrix}
-9 & 6 & | & 0 \\
-18 & k & | & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -\frac{2}{3} & | & 0 \\
-18 & k & | & -3
\end{bmatrix}
\]

\[
R_2 + 18R_1 \rightarrow \begin{bmatrix}
1 & -\frac{2}{3} & | & 0 \\
0 & k-12 & | & -3
\end{bmatrix}
\]

To be consistent \[k-12 \neq 0\] so \[k \neq 12\]

(2) (5 pts) Suppose $\alpha$ is a real number. The solution(s) for the system of linear equations corresponding to the following augmented matrix

\[
\begin{bmatrix}
1 & 0 & 7 & | & 1 \\
0 & 1 & -2 & | & 2 \\
0 & 0 & 0 & | & \alpha
\end{bmatrix}
\]

is (are):

(Hint: Your answer will depend on the value of $\alpha$.)

If \[\alpha \neq 0\] the system is inconsistent so there are no solutions.

If \[\alpha = 0\] the system is consistent and there are infinitely many solutions.

\[
\begin{cases}
(x = 1 - 7t, \ y = 2 + 2t, \ z = t)
\end{cases}
\]

pts: /10
solutions: \( \{(3-2t, -4+5t, t) \mid t \in \mathbb{R}\} \) \( \mathbb{R}^3 \) 

7. (11 pts) Solve the following system of linear equations

\[
\begin{align*}
2x + y - z &= 2 \\
4x + y + 3z &= 8 \\
8x + 3y + z &= 12
\end{align*}
\]

by writing the corresponding augmented matrix and then by row reducing.

\[
\begin{bmatrix}
2 & 1 & -1 & 2 \\
4 & 1 & 3 & 8 \\
8 & 3 & 1 & 12
\end{bmatrix} \xrightarrow{\frac{1}{2}R_1}
\begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & 1 \\
4 & 1 & 3 & 8 \\
8 & 3 & 1 & 12
\end{bmatrix}
\]

\( R_2 - 4R_1 \)

\[
\begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & 1 \\
0 & -1 & 5 & 4 \\
0 & 0 & 9 & 8
\end{bmatrix} \xrightarrow{-R_2}
\begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & 1 \\
0 & 1 & -4 & -4 \\
0 & 0 & 9 & 8
\end{bmatrix}
\]

\( R_3 - 8R_1 \)

\[
\begin{bmatrix}
1 & \frac{1}{2} & -\frac{1}{2} & 1 \\
0 & 1 & -4 & -4 \\
0 & 0 & 9 & 8
\end{bmatrix} \xrightarrow{R_1 - \frac{1}{2}R_2}
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -4 & -4 \\
0 & 0 & 9 & 8
\end{bmatrix}
\]

\( R_3 + R_2 \)

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -4 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(4 pts) How many solution(s) does the system have? Write it (them) out explicitly.

The system reduces to

\[
\begin{align*}
x + 2z &= 3 \\
y - 5z &= -4
\end{align*}
\]

Thus, \( z \) can be any value \( t \in \mathbb{R} \)

\[
x = 3 - 2t \\
y = -4 + 5t \\
z = t
\]

There are infinite sol.

pts: 15
8. Let \( A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix} \).

(1) (5 pts) Compute the product \( A^T A \).

\[
A^T = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}
\]

\[
A^T A = \begin{bmatrix} 13 & -3 & -4 \\ -3 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}
\]

(2) (5 pts) Compute the product \( AA^T \).

\[
A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & 2 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}
\]

\[
AA^T = \begin{bmatrix} 14 & 4 \\ 4 & 5 \end{bmatrix}
\]

pts: /10
9. (1) (5 pts) Which of \[
\begin{bmatrix}
-19 & -2 & 11 \\
9 & 1 & -5 \\
2 & 0 & 1
\end{bmatrix}
\] and \[
\begin{bmatrix}
13 & -2 & -5 \\
-7 & 1 & 3 \\
2 & 0 & -1
\end{bmatrix}
\] is the inverse of \[
\begin{bmatrix}
1 & 2 & 1 \\
1 & 3 & 4 \\
2 & 4 & 1
\end{bmatrix}
\]?

\[
\begin{bmatrix}
13 & -2 & -5 \\
-7 & 1 & 3 \\
2 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 \\
3 & 4 & 1 \\
2 & 4 & 1
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(2) (5 pts) Use the information found in (1) to solve the following system of linear equations, by writing it in matrix form \(AX = B\). Explain what you are doing.

\[
\begin{align*}
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 4x_3 &= 6 \\
2x_1 + 4x_2 + x_3 &= 5
\end{align*}
\]

Write the system as

\[
\begin{bmatrix}
1 & 2 & 1 \\
1 & 3 & 4 \\
2 & 4 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}
\]

Multiply both sides by the inverse:

\[
A^{-1} \cdot A \cdot \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
13 & -2 & -5 \\
-7 & 1 & 3 \\
2 & 0 & -1
\end{bmatrix}
\begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}
\]

pts: /10
**Bonus. (Infectious disease dynamics)** The spread of an infectious disease, such as influenza, is often modeled using the following autonomous differential equation

\[ \frac{dI}{dt} = \beta I(N - I) - \mu I, \]

where \( I \) is the number of infected people, \( N \) is the total size of the population being modeled, \( \beta \) is a constant determining the rate of transmission, and \( \mu \) is the rate at which people recover from infection.

(a) (5 pts) Suppose \( \beta = 0.01 \), \( N = 1000 \), and \( \mu = 2 \). That is, \( \frac{dI}{dt} = 8I - 0.01I^2 \).

Find all equilibria and determine whether each is stable or unstable.

How many people will be infected in the long run?

(b) (5 pts) Leaving the constants unspecified, what are the equilibria of the model in terms of these constants? Determine whether each is stable or unstable in terms of a condition on the difference \( \beta N - \mu \).
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Find all equilibria and determine whether each is stable or unstable.

How many people will be infected in the long run?

\[
\frac{dI}{dt} = 0.01(1000 - I) - 2I \quad \Rightarrow \quad \frac{dI}{dt} = 10I - 0.01I^2 - 2I
\]

\(\Rightarrow \frac{dI}{dt} = 8I - 0.01I^2 \quad \text{The equilibria are given by} \quad 8I - 0.01I^2 = 0 \quad \Rightarrow \quad I(8 - 0.01I) = 0 \quad \Rightarrow \quad \hat{I} = 0, \hat{I} = 800\)

(b) (5 pts) Leaving the constants unspecified, what are the equilibria of the model in terms of these constants? Determine whether each is stable or unstable in terms of a condition on the difference \(\beta N - \mu\).

\(\hat{I} = 800\) is locally stable in the long run: there will be 800 people infected.

\(\hat{I} = 0\) is unstable.
In general

\[ \beta I (N-I) - \mu I = 0 \]

\[ \iff \quad I (\beta N - \mu - \beta I) = 0 \]

\[ \implies \hat{I} = 0 \quad \text{or} \quad I = \frac{\beta N - \mu}{\beta} \]

\[ g(N) = (\beta N - \mu) I - \beta I^2 \]

\[ g'(N) = \beta N - \mu - 2\beta I \]

\[ g'(0) = \beta N - \mu \]

\[ g' \left( \frac{\beta N - \mu}{\beta} \right) = (\beta N - \mu - 2\beta) \cdot \frac{\beta N - \mu}{\beta} = - (\beta N - \mu) \]

If \( \beta N - \mu > 0 \) then

\[ \hat{I} = 0 \] is unstable

\[ \hat{I} = \beta N - \mu \]

is locally stable
\[ \text{If } \beta N - \mu < 0 \quad \text{then} \]
\[ \hat{I} = 0 \quad \text{is locally stable} \]
\[ \hat{I} = \frac{\beta N - \mu}{\beta} \quad \text{is unstable} \]

\[ \beta N - \mu > 0 \]

\[ \frac{d\hat{I}}{dt} \]

\[ \hat{I} \]

\[ \beta N - \mu < 0 \]

\[ \frac{d\hat{I}}{dt} \]

\[ \hat{I} \]

Unrealistic because there would be a negative number of infective people.

In reality this would say that \( \mu \) is much larger than \( \beta \) = infection rate.