Lee, Chapter 4

Exercise 10 Let $S$ be the square $I \times I$ with the order topology generated by the dictionary topology.

(a) Show that $S$ has the least upper bound property.

(b) Show that $S$ is connected.

(c) Show that $S$ is locally connected but not locally path-connected.

Proof. (a) Let $A \subseteq S$. For each $a \in A$, $(0, 0) \leq a \leq (1, 1)$, so $A$ is bounded (above). Now, for each $t \in I$, if $(I \times \{t\}) \cap A \neq \emptyset$, then $(I \times \{t\}) \cap A$ has a least upper bound $\alpha_t$. Similarly, for each $t \in I$, if $(\{t\} \times I) \cap A \neq \emptyset$ then $(\{t\} \times I) \cap A$ has a least upper bound $\beta_t$. Now, $\{\alpha_t\}_{t \in I}$ and $\{\beta_t\}_{t \in I}$ are bounded subsets of $\mathbb{R}$, so they have least upper bounds $\alpha$ and $\beta$, respectively. It follows that for each $(a_1, a_2) \in A$, $a_1 \leq \alpha$ and $a_2 \leq \beta$, so $(a_1, a_2) \leq (\alpha, \beta)$, making $(\alpha, \beta)$ an upper bound of $A$. Now, suppose $(\gamma, \delta) < (\alpha, \beta)$ is an upper bound of $A$. Then for all $(a_1, a_2) \in A$, $a_1 \leq \gamma < \alpha$ or $\gamma = \alpha$ and $a_2 \leq \delta < \beta$. But, since $\alpha$ is the least upper bound of the “$x$-coordinates” of $A$ and $\beta$ is the least upper bound of the “$y$-coordinates” of $A$, both of these statements are contradictions. As a result, $(\gamma, \delta) \geq (\alpha, \beta)$ for all upper bounds $(\gamma, \delta)$ of $A$, making $(\alpha, \beta)$ the least upper bound of $A$ by definition.

(b) Suppose $S$ is not connected. Then there exist disjoint, nonempty open subsets $H$ and $K$ such that $H \cup K = S$. Without loss of generality, assume $(1, 1) \in K$. Now, $K$ contains some neighborhood of $(1, 1)$, so $(\alpha, \beta) = \sup(H) \neq 1$. Since $H \cup K = S$ and $H$ and $K$ are both open, $(\alpha, \beta)$ is an element of some neighborhood which is a subset of either $H$ or $K$. But, any neighborhood of $(\alpha, \beta)$ contains a set of the form $\{s \in S \mid r < s < t\}$ for some $r, t \in S$, which intersects both $H$ and $K$. This is a contradiction, so $S$ is connected.

(c) For any $(s_1, s_2) \in S$, the closed neighborhoods $\{(t_1, t_2) \in S \mid (s_1 - \frac{1}{n}, s_2 - \frac{1}{n}) \leq (t_1, t_2) \leq (s_1 + \frac{1}{n}, s_2 + \frac{1}{n})\}_{n \in \mathbb{N}}$ of $s$ are homeomorphic to either $I$ or $S$, each of which is connected. Consequently, each neighborhood of $s$ contains a connected neighborhood of $s$, making $S$ locally connected.
Exercise 11 Let $X$ be a topological space, and let $C(X)$ be the cone on $X$.

(a) Show that $C(X)$ is path-connected.

(b) Show that $C(X)$ is locally (path)-connected iff $X$ is.

Proof. (a) Let $\{C_\alpha\}_{\alpha \in A}$ be the path-components of $X$. Notice that each $C(C_\alpha)$ is a path-connected subspace of $C(X)$. Now, let $\xi$ be the “tip” of the cone, and let $x, y \in C(X)$. If $x, y \in C_\alpha$, then the proof is trivial, so assume $x \in C(C_\alpha)$ and $y \in C(C_\beta)$, where $\alpha \neq \beta$. Then there exists a path $\gamma : I \to C(C_\alpha)$ such that $\gamma(0) = x$ and $\gamma(1) = \xi$, and a path $\delta : I \to C(C_\beta)$ such that $\delta(0) = \xi$ and $\delta(1) = y$. If we concatenate these paths, we get a path $\epsilon : I \to C(X)$, where $\epsilon(t) = \gamma(2t)$ when $0 \leq t \leq \frac{1}{2}$, and $\epsilon(t) = \delta(2t - 1)$ when $\frac{1}{2} \leq t \leq 1$. Hence, $C(X)$ is path-connected.

(b) First we will prove that $C(X)$ is locally connected iff $X$ is.

($\Rightarrow$)
Exercise 13 Let $T$ be the topologist’s sine curve.

(a) Show that $T$ is connected but not path-connected or locally connected.

(b) Determine the components and the path-components of $T$.

Proof. (a) Suppose $T$ is disconnected. Then there exist disjoint, nonempty open subsets $H$ and $K$ such that $H \cup K = T$. Notice that $T_0$ and $T_+$ are both connected and path-connected, so there is no disjoint union of open sets equal to either piece of $T$. Thus, without loss of generality, suppose $T_0 \subset H$. But any open set which contains $T_0$ also contains an open subset $H_{T_+}$ of $T_+$ such that $H_{T_+} \cup K = T_+$. This is a contradiction, so $T$ is connected. However, suppose a path $\gamma : I \to T$ exists such that $\gamma(0) = (\frac{1}{2\pi}, 0)$ and $\gamma(1) = (0, 0)$. Now, consider the point $\tau = \inf\{t \in I \mid \gamma(t) \in T_0\}$. It follows that the image $\gamma([0, \tau])$ has at most one element of $T_0$, but notice that $T_0 \subseteq \overline{\gamma([0, \tau])}$, so $\gamma([0, \tau]) \neq \overline{\gamma([0, \tau])}$ and thus $\gamma([0, \tau])$ is not closed. But $\gamma([0, \tau])$ is the continuous image of a compact set of $\mathbb{R}$, which is closed in $\mathbb{R}$. This is a contradiction, so no such path can exist and $T$ is not path-connected. Finally, $T$ is not locally connected, because the intersection of $T$ and the ball of radius $\frac{1}{2}$ contains no connected neighborhood.

(b) Since $T$ is connected, $T$ has only one component - namely, $T$ itself. Now, notice in the discussion above that the non-existence of the path $\gamma$ did not depend on the exact location of the point in $T_+$; it only depended on the fact that $(\frac{1}{2\pi})$ was in $T_+$ and not in $T_0$. Now, $T_0$ and $T_+$ are path-connected and, in fact, they are the path-components of $T$. ■
Exercise 15 Suppose $G$ is a topological group.

(a) Show that every open subgroup of $G$ is also closed.

(b) For any neighborhood $U$ of $1$, show that the group $\langle U \rangle$ generated by $U$ is open and closed in $G$.

(c) For any connected subset $U \subseteq G$ containing $1$, show that $\langle U \rangle$ is connected.

(d) Show that if $G$ is connected, then every connected neighborhood of $1$ generates $G$.

Proof. (a) Suppose $H \subseteq G$ is an open subgroup, and consider $G - H$. This is equal to $\bigcup_{g \in G - H} (gH)$. Further, since $H$ is open and left translation is a homeomorphism, each $gH$ is open, so $G - H$ is open. Therefore, $H$ is closed.

(b) The group $\langle U \rangle$ is equal to $\bigcup_{g \in G} (gU)$, which is an open subgroup, so by (a), $\langle U \rangle$ is open and closed in $G$. 

\[ \blacksquare \]
Willard, Section 23

Problem A

Exercise 1 Prove that the looped line is metrizable.
Proof. By exercise 14A(4), the looped line is $T_{3.5}$, so it is $T_1$ and (completely) regular. Further, $\mathcal{B} = \{(x-q, x+q) \mid x \in \mathbb{Q} - \{0\}, q \in \mathbb{Q}\} \cup \{(-\infty, -n) \cup (-q, q) \cup (n, \infty) \mid q \in \mathbb{Q}, n \in \mathbb{N}\}$ is a countable basis for the looped line, so it is second countable. Finally, by Urysohn’s Metrization Theorem, the slotted line is metrizable. ■

Exercise 2 Prove that the scattered line is not metrizable.
Proof. Clearly the scattered line is $T_1$, but it is not second countable, since the irrational numbers with the discrete metric have no countable basis. Hence, by the Urysohn Metrization Theorem, the scattered line is not metrizable. ■

Exercise 3 Prove that the disjoint union of metrizable spaces is metrizable.
Proof. Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of metrizable spaces. Construct metrics $\rho_\alpha$ bounded above by 1 for each space $X_\alpha$. Now, define a metric $\rho$ on $\bigsqcup_{\alpha \in A} X_\alpha$ as follows: $\rho(x, y) = \rho_\alpha(x, y)$ if $x, y \in X_\alpha$ and $\rho(x, y) = 1$ if $x \in X_\alpha, y \in X_\beta, \alpha \neq \beta$. The metric $\rho$ generates the topology of $\bigsqcup_{\alpha \in A} X_\alpha$, so it is metrizable. ■

Exercise 4 Let $A$ be an infinite set, and let $X$ be the “hedgehog space” of spininess $|A|$. Does the metric

$$\rho(x, y) = |x - a| + |a - y|, x \in I_\alpha, y \in I_\beta, \alpha \neq \beta$$
$$\rho(x, y) = |x - y|, x, y \in I_\alpha,$$

where $a$ is the common point of each $I_\alpha$, generate the topology of $X$?
Proof. An open subset of $X$ is a union of open subsets $\{U_\beta\}_{\beta \in B}$, where each $U_\beta$ is open in some $I_\alpha$. Thus, each $U_\beta$ is of the form $[a, x)$, $(a, b)$, or $(b, 1]$. Each of these sets can be expressed as an open set in the set $X$ with the metric $\rho$, so $\rho$ generates the topology on $X$. ■
Problem C

Prove that, for a locally compact space $X$, the following are equivalent:

(a) $X$ is separable.

(b) $X = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ is compact and $K_n \subset (K_{n+1})^\circ$.

(c) The one-point compactification $X^*$ of $X$ is metrizable.

Proof.
Problem B

Exercise 1 Prove that the continuous image of a path-connected space is path-connected.

Proof. Let $f : X \to Y$ be continuous, and let $X$ be path-connected. Without loss of generality, we can assume that $f$ is surjective. Then for distinct $y_1, y_2 \in Y$, there are points $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $X$ is path-connected, there is a path $\gamma : I \to X$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Now, because $\gamma$ is continuous, the composition $f \circ \gamma : I \to Y$ is also continuous. Further, $f(\gamma(0)) = f(x_1) = y_1$ and $f(\gamma(1)) = f(x_2) = y_2$. Hence, $f \circ \gamma$ is a path from $y_1$ to $y_2$, and therefore $Y$ is path-connected.

Exercise 2 Prove that the nonempty product of finitely many spaces is path-connected iff each factor space is connected.

Proof. ($\Rightarrow$) Suppose $X_1, X_2, \ldots, X_n$ are topological spaces, and $X = \prod_{i=1}^{n} X_i$ is path-connected. Let $i, 1 \leq i \leq n$ be arbitrary, and consider distinct points $x_i, y_i \in X_i$. Then $x = (x_1, x_2, \ldots, x_i, \ldots, x_n)$ and $y = (x_1, x_2, \ldots, y_i, \ldots, x_n)$ are distinct points in $X$ for arbitrary $x_j \in X_j, j \neq i$, so there is a path $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since $\pi_i$ is continuous, so too is $\pi_i \circ \gamma : I \to X_i$. Further, $\pi_i(\gamma(0)) = x_i$ and $\pi_i(\gamma(1)) = y_i$, making $\pi_i \circ \gamma$ a path from $x_i$ to $y_i$. Because $i, x_i$, and $y_i$ were arbitrary, it follows that each $X_i$ is path-connected.

($\Leftarrow$) Suppose each $X_1, X_2, \ldots, X_n$ is path-connected. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be distinct points in $X$. Now, $\{a_1\} \times \{a_2\} \times \ldots \times X_i \times \ldots \times \{a_n\}$ is homeomorphic to $X_i$ for each $i$ and for each $a = (a_1, a_2, \ldots, a_n) \in X$, and path-connectedness is preserved by homeomorphism. In short, there exist paths $\gamma_i : I \to (\{y_1\} \times \{y_2\} \times \ldots \times \{y_{i-1}\} \times X_i \times \{x_{i+1}\} \ldots \times \{x_n\})$ such that $\gamma(0) = (y_1, y_2, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $\gamma(1) = (y_1, y_2, \ldots, y_{i-1}, y_i, x_{i+1}, \ldots, x_n)$. Because $X$ is a finite product, we can concatenate the $\gamma_i$'s into a well-defined, continuous path $\gamma : I \to X$ from $x$ to $y$, proving that $X$ is path-connected.