Exercise 1  Define \( f : (−\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \) by \( f(0) = 0 \), and \( f(x) = \frac{x−\sin(x)}{1−\cos(x)} \) if \( x \neq 0 \).

(a) Show that \( f \) is continuous at \( x = 0 \).

(b) Show that \( f \) is strictly increasing, and that the image of the interval is all of \( \mathbb{R} \).

Proof. (a) To prove that \( f \) is continuous at 0, it will suffice to use L'Hôpital's Rule to show that \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0 \). Let \( g(x) = x − \sin(x) \), and \( h(x) = 1 − \cos(x) \). Then \( g \) and \( h \) are both twice differentiable for all \( x \in \mathbb{R} \), where \( g'(x) = 1 − \cos(x) \), \( g''(x) = \sin(x) \), \( h'(x) = \sin(x) \), and \( h''(x) = \cos(x) \). Now, notice that for each \( x \in (−\frac{\pi}{2}, 0) \), \( h(x) \neq 0 \) \( \neq h'(x) \) \( \neq 0 \) \( \neq h''(x) \), \( \lim_{x \to 0^-} g(x) = 0 = \lim_{x \to 0^+} h(x) \), and \( \lim_{x \to 0^-} g'(x) = 0 = \lim_{x \to 0^+} h'(x) \), so by two iterations of L'Hôpital's Rule,

\[
\lim_{x \to 0^-} \frac{x − \sin(x)}{1 − \cos(x)} = \lim_{x \to 0^-} \frac{1 − \cos(x)}{\sin(x)} = \lim_{x \to 0^-} \frac{\sin(x)}{\cos(x)}.
\]

Now, \( \frac{\sin(x)}{\cos(x)} \) is continuous at \( x = 0 \), and \( \frac{\sin(0)}{\cos(0)} = 0 \), so

\[
\lim_{x \to 0^-} \frac{x − \sin(x)}{1 − \cos(x)} = 0.
\]

A similar argument on \( (0, \frac{\pi}{2}) \) gives us

\[
\lim_{x \to 0^+} \frac{x − \sin(x)}{1 − \cos(x)} = 0.
\]

(b) Note that since \( f(−x) = \frac{(−x)−\sin(−x)}{1−\cos(−x)} = −\frac{x−\sin(x)}{1−\cos(x)} = −f(x) \), \( f \) is an odd function. Now, let \( A = (−a, a) \subseteq \mathbb{R} \), and let \( \omega : A \to \mathbb{R} \) be an odd function. Then for \( \alpha, \beta \in A \), if \( \alpha < \beta \leq 0 \) implies \( \omega(\alpha) < \omega(\beta) \), then \( \omega(−\beta) = −\omega(\beta) < −\omega(\alpha) = \omega(−\alpha) \). In short, if \( \omega \) is strictly increasing on \( (−a, 0] \), then it is increasing on \( [0, a] \) and thus \((−a, a) \). Hence, it suffices to show that \( f \) is strictly increasing on the interval \( (−\frac{\pi}{2}, 0] \). Consider the numerator \( g(x) = x − \sin(x) \) and the denominator \( h(x) = 1 − \cos(x) \). Note that on \( (−\frac{\pi}{2}, 0) \), \( g'(x) = 1 − \cos(x) > 0 \), so by Page 104 Exercise 5, \( g \) is strictly increasing. Further, \( h'(x) = \sin(x) < 0 \), so \( h \) is decreasing. As a result, \( f = \frac{g}{h} \) is increasing on \( (−\frac{\pi}{2}, 0) \). Finally, note that \( f(0) = 0 \), and if \( c < 0 \), \( f(c) < 0 = f(0) \), so \( f \) is strictly increasing on \( (−\frac{\pi}{2}, 0] \). To prove that the image of \( (−2\pi, 2\pi) \) is \( \mathbb{R} \), note that \( \lim_{x \to 2\pi^-} f(x) = +\infty \) and \( \lim_{x \to −2\pi^+} f(x) = −\infty \), and since \( f \) is increasing on this interval, \( f((−\pi, \pi)) = (−\infty, \infty) = \mathbb{R} \). \( \blacksquare \)
Exercise 1  (a) Suppose $f$ is a continuous, real-valued function on the interval $[a,b]$. Prove that there exists $c \in [a,b]$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Proof. Let $F(x) = \int_a^x f(t)\,dt$. Now, $F$ is continuous and differentiable on $[a,b]$, so by the Mean Value Theorem, there is a $c \in (a,b)$ such that $\frac{F(b) - F(a)}{b-a} = F'(c)$ or, equivalently, $F(b) - F(a) = F'(c)(b-a)$. But, by the Fundamental Theorem of Calculus, $F(b) - F(a) = \int_a^b f(x)\,dx$, and by differentiation of the integral, $F'(c) = f(c)$, so we get $\int_a^b f(x)\,dx = f(c)(b-a)$, thus completing the proof. ■