Chapter 6

Exercise 2 Show that it is possible to obtain a space homeomorphic to a projective plane $\mathbb{P}^2$ by attaching a disk $\bar{B}^2$ to a M"obius band $\mathbb{M}$ along their boundaries.

Proof. Although technically it breaks a rule, we can consider a polygonal presentation of the form $\langle a, b \mid ab \rangle$ to be that of the disk, since no identifications are made on the boundary. Now, recall that a standard polygonal presentation of $\mathbb{P}^2$ is $\langle a \mid aa \rangle$. From the elementary operations of polygonal presentations, we see

$$\langle a \mid aa \rangle \approx \langle b, c, d \mid bcdbcd \rangle$$

$$\approx \langle a, b, c, d, e \mid bcdbe^{-1}a^{-1}, accd \rangle$$

$$\approx \langle b, f, g \mid bfbg^{-1}, gf \rangle,$$

which is a polygonal presentation of a disk and a M"obius band attached solely and entirely along their boundaries. Since we have shown that a polygonal presentation of $\mathbb{P}^2$ is topologically equivalent to that of $\bar{B}^2$ and $\mathbb{M}$ attached along their boundaries, it follows that these spaces are homeomorphic. ■

Exercise 3 Show that the Klein bottle $\mathbb{K}$ is homeomorphic to a quotient obtained by attaching two M"obius bands together along their boundaries.

Proof. Recall from Lemma 6.16 that $\mathbb{K} \cong \mathbb{P}^2 \# \mathbb{P}^2 \cong \langle a, b \mid aabb \rangle$. Using the elementary operations of polygonal presentations, we find that

$$\langle a, b \mid aabb \rangle \approx \langle a, b, c \mid acacbb \rangle$$

$$\approx \langle a, b, c, d, e \mid acad, d^{-1}cbb \rangle$$

$$\approx \langle a, b, c, d, e \mid acad, bd^{-1}e, e^{-1}cb \rangle$$

$$\approx \langle a, b, c, d, e \mid acad, e^{-1}cb, b^{-1}e^{-1}d \rangle$$

$$\approx \langle a, c, d, e \mid acad, e^{-1}ce^{-1}d \rangle,$$

which is a polygonal presentation of two M"obius bands attached solely and entirely along their boundaries. Hence, a polygonal presentation of $\mathbb{K}$ is topologically equivalent to that of two M"obius bands attached along their boundaries, so the spaces are homeomorphic. ■