Chapter 7

Exercise 2 Suppose $X$ is a topological space and $g$ is any path in $X$ from $p$ to $q$. Let $\Phi_g : \pi_1(X, p) \to \pi_1(X, q)$ be the group isomorphism defined by $\Phi_g[\gamma] = [g] \cdot [\gamma] \cdot [g]$.

(a) Show that if $h$ is another path in $X$ starting at $q$, then $\Phi_{g \cdot h} = \Phi_h \circ \Phi_g$.

(b) Suppose $\psi : X \to y$ is continuous and show that $\Phi_{\psi \circ g} \circ \psi_* = \psi_* \circ \Phi_g$.

Proof.

(a) By algebra and what we know about path composition, for any $[\gamma] \in \pi_1(X, p)$,

$$
\Phi_{g \cdot h}[\gamma] = [g \cdot h] \cdot [\gamma] \cdot [g \cdot h] \\
= [h \cdot g] \cdot [\gamma] \cdot [g \cdot h] \\
= [h] \cdot [g] \cdot [\gamma] \cdot [g] \cdot [h] \\
= [h] \cdot \Phi_g[\gamma] \cdot [h] \\
= \Phi_h(\Phi_g[\gamma]) = (\Phi_h \circ \Phi_g)[\gamma],
$$

so $\Phi_{g \cdot h} = \Phi_h \circ \Phi_g$.

(b) Recalling that $\psi_*[\gamma] = [\psi \circ \gamma]$, for any $[\gamma] \in \pi_1(X, p)$ we have

$$
(\Phi_{\psi \circ g} \circ \psi_*)[\gamma] = \Phi_{\psi \circ g}[\psi \circ \gamma] \\
= [\psi \circ g] \cdot [\psi \circ \gamma] \cdot [\psi \circ g] \\
= [\psi \circ g] \cdot [\psi \circ \gamma] \cdot [\psi \circ g] \\
= \psi_*[g] \cdot \psi_*[\gamma] \cdot \psi_*[g],
$$

which, since $\psi_*$ is a group homomorphism, gives us

$$
\psi_*[g \cdot \gamma \cdot g] = \psi_*([g] \cdot [\gamma] \cdot [g]) = \psi_*(\Phi_g[\gamma]) = (\psi_* \circ \Phi_g)[\gamma],
$$

so $\Phi_{\psi \circ g} \circ \psi_* = \psi_* \circ \Phi_g$. 

$\blacksquare$
Exercise 3 Let $X$ be a path-connected topological space and let $p, q \in X$. Show that all paths from $p$ to $q$ give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$ if and only if $\pi_1(X, p)$ is abelian.

Proof.

$(\Rightarrow)$ Suppose all paths from $p$ to $q$ give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$. Then for path $f$ from $p$ to $q$ and $[\gamma], [\delta] \in \pi_1(X, p)$,

$\Phi_{\gamma f}( [\gamma] \cdot [\delta] ) = [\gamma \cdot f] \cdot [\gamma] \cdot [\delta] \cdot [\gamma \cdot f]$

$= [f] \cdot [\gamma] \cdot [\gamma] \cdot [\delta] \cdot [\gamma] \cdot [f]$

$= [f] \cdot [\gamma] \cdot [\gamma] \cdot [\delta] \cdot [\gamma] \cdot [f]$

$= [f] \cdot [\delta] \cdot [\gamma] \cdot [f]$

$= \Phi_f([\delta] \cdot [\gamma]).$

But, since $\Phi_{\gamma f} = \Phi_f$ by assumption, and since both are bijections, we have that $[\gamma] \cdot [\delta] = [\delta] \cdot [\gamma]$, making $\pi_1(X, p)$ abelian.

$(\Leftarrow)$ Suppose $\pi_1(X, p)$ is abelian. Then for paths $f, g$ from $p$ to $q$ and $[\gamma] \in \pi_1(X, p)$, we have $[\gamma] = [\gamma] \cdot [g] \cdot [f] \cdot [g]$. Since $\pi_1(X, p)$ is abelian, and $[f] \cdot [g] \cdot [\gamma]$ and $[g] \cdot [f]$ are all in $\pi_1(X, p)$, we have $[\gamma] = [f] \cdot [g] \cdot [\gamma] \cdot [g] \cdot [f]$. But this implies that $[f] \cdot [\gamma] \cdot [f] = [g] \cdot [\gamma] \cdot [g]$ or, equivalently, $\Phi_f = \Phi_g$. Because $f, g,$ and $[\gamma]$ were all arbitrary, it follows that all paths from $p$ to $q$ give the same isomorphism of $\pi_1(X, p)$ with $\pi_1(X, q)$.

Exercise 6 For any path-connected space $X$ and any base point $p \in X$, show that the map sending a loop to its circle representative induces a bijection between the set of conjugacy classes of elements of $\pi_1(X, p)$ and $[S^1, X]$ (the set of free homotopy classes of continuous maps from $S^1$ to $X$).

Proof. ■
Exercise 9 Suppose $X$ and $Y$ are connected topological spaces and the fundamental group of $Y$ is abelian. Show that if $F, G : X \to Y$ are homotopic maps such that $F(x_0) = G(x_0)$ for some $x_0 \in X$, then $F_* = G_* : \pi_1(X, x_0) \to \pi_1(Y, F(x_0))$. Give a counterexample to show that this might not be true if $\pi_1(Y)$ is not abelian.

Proof. Call $F(x_0) = y_0$ and assume that there is a homotopy $H : X \times I \to Y$ from $F$ to $G$ such that $H(x_0, t) = y_0$ for all $t \in I$. Then for a loop $\gamma$ whose base point is $x_0$, we have that $H(\gamma(s), 0) = (F \circ \gamma)(s)$, $H(\gamma(s), 1) = (G \circ \gamma)(s)$, and $H(\gamma(0), t) = H(\gamma(1), t) = y_0$ for all $t \in I$, so we know $F_*[\gamma] = [F \circ \gamma] = [G \circ \gamma] = G_*[\gamma]$, making $F_* = G_*$. I assume that the fact $\pi_1(Y, y_0)$ is abelian was swept up somewhere in our initial assumption about $H$. I have no counterexample to reinforce the necessity of this assumption.

Exercise 11 Show that the Möbius band is homotopy equivalent to $\mathbb{S}^1$.

Proof. We will show that $\mathbb{S}^1$ is a deformation retract of the Möbius band. First, consider $I \times I$ and the subspace $S = \{(x, \frac{1}{2}) \mid x \in I\}$. Surely the function $r : I \times I \to S$ defined by $r(x, y) = (x, \frac{1}{2})$ is a retraction of $I \times I$ onto $S$. Further, $\iota_S \circ r$ is homotopic to $\text{Id}_{I \times I}$, where $\iota_S$ is the inclusion of $S$ into $I \times I$, since $I \times I$ is a convex subset of $\mathbb{R}^2$. Hence, $r$ is a deformation retraction, and $S$ is a deformation retract of $I \times I$ (see figure). Now, if we consider the identification space $I \times I/\sim$, where $(x, y)$ is identified with itself when $x \in (0, 1)$ and $(0, y) \sim (1, 1 - y)$ for all $y \in I$, then we have the Möbius band. Additionally, if we apply this quotient map to $S$, the only identification made is $(0, \frac{1}{2}) \sim (1, \frac{1}{2})$, and since $S$ is homeomorphic to $I$, we get $\mathbb{S}^1$. Since quotient maps preserve deformation retracts, we have that $\mathbb{S}^1$ is a deformation retract of the Möbius band. Hence, by Proposition 7.46, the two spaces are homotopy equivalent.
Exercise 15 Let $X$ be the union of the three circles with radius 1 and centers at $(0,0)$, $(2,0)$, and $(4,0)$. Prove that $X$ is homotopy equivalent to a bouquet of three circles.

**Proof.** We will show that both spaces are deformation retracts of $\mathbb{R}^2$ minus three distinct points, thus concluding by Proposition 7.46 that the spaces are homotopy equivalent. Considering $X$ as a subspace of $\mathbb{R}^2$, start by removing the centers of the circles. Then, as described for the figure-eight space, we define the deformation retraction by “...carving the space up into regions in which straight-line homotopies are easily defined...” namely, the insides of the circles (minus their centers) and the space on the “outside” of $X$. “The resulting maps are continuous by the gluing lemma.” Hence, $X$ is a deformation retract of $\mathbb{R}^2$ minus three points. Similarly, for the bouquet of three circles, remove the centers of the circles and construct a similar deformation retraction onto the bouquet (see figure).